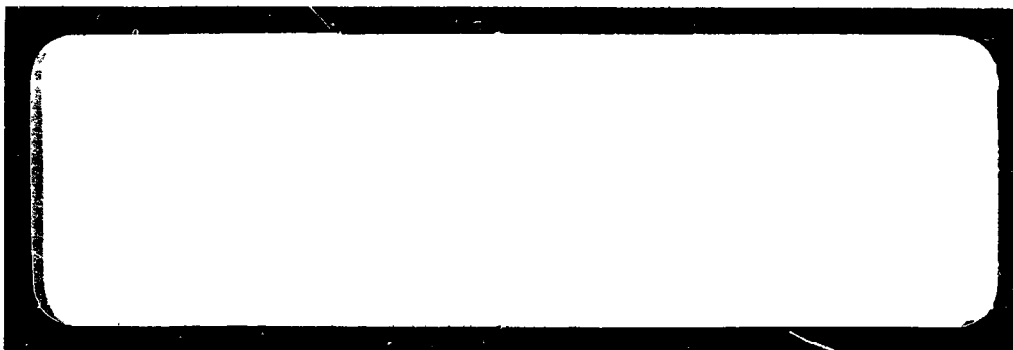


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**COMMERCIAL  
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AN ANALYSIS OF METHODS FOR  
PREDICTING THE STABILITY CHARACTERISTICS  
OF AN ELASTIC AIRPLANE

SUMMARY REPORT

By

Members of the Aerodynamics  
and Structures Research Organizations

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November 1968

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## 1. SUMMARY

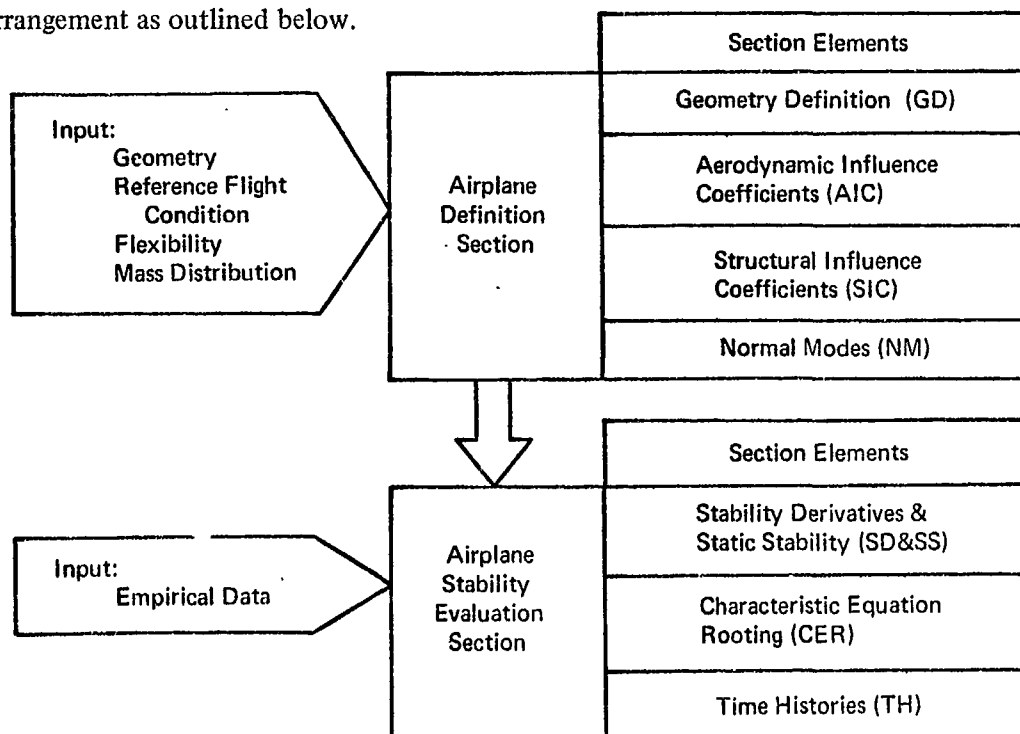
This report describes an investigation that was conducted to select and develop an improved analytical method for predicting the stability and control characteristics of elastic airplanes.

The investigation included consideration of the free-flight conditions of large airplanes in their "clean" configuration. Landing, takeoff, ground effects, stability augmentation, and control surface movements were not considered, nor were prediction of airplane performance, flutter, or structural loads. The study was confined to a flight envelope extending from low subsonic speeds to Mach 5 and from sea level to 30 000 meters (93 360 ft) altitude. Within this envelope, rectilinear and curvilinear reference flight paths were taken and analyses made of arbitrary, large, and small perturbations of airplane motion about the reference flight paths. Airplane structural motions of dynamic-elastic and quasi-static-elastic characters were included, and their effects on stability determined.

The approach taken in the investigation was to develop in order:

- equations of motion
- stability criteria
- stability derivative prediction methods
- stability characteristics prediction methods

A major result obtained in the investigation has been the unification and development of aerodynamic, structural, and dynamic technologies into an overall plan for calculating elastic airplane stability characteristics. The plan involves a computing program system arrangement as outlined below.



There are two essential features to this plan:

1. The system elements and the total system are well suited to computer programming.
2. Provision is made for introducing empirical data (experimental measurements, handbook results, etc.) into the system.

Other results and conclusions concerning the analysis methods and their application to large, flexible airplanes of the SST and 707-320B type are listed below.

1. The lumped parameter concept is the most practical way to represent an airplane for analysis. This involves paneling the airplane so that the aerodynamic and structural relationships among panels can be expressed in terms of influence coefficients. A geometry definition program is desirable to mechanize and help guide the paneling.
2. Lifting surface theory gives good results and is the most suitable method for determining the aerodynamic influence coefficients.
3. For most airplane configurations, equivalent beam structural models are adequate for determining structural influence coefficients. For very low aspect ratio wings or for increased accuracy, however, influence coefficients from large, sophisticated, finite element structural programs should be used.
4. The stability criteria that apply to rigid airplanes apply directly to elastic airplanes.
5. Stability characteristics can be influenced strongly by inaccuracies in estimating the rigid stability derivatives. These effects on design decisions are of the same order of importance as quasi-static-elastic and dynamic-elastic effects.
6. The effect of airplane loading conditions (mass distribution) on an elastic airplane's stability characteristics can be relatively large, sometimes reversing the sign of the parameters that represent flexibility effects.
7. The evaluations of control effectiveness and control surface angles required to trim are influenced strongly by flexibility; however, since viscous and nonlinear aerodynamic effects are also important in those evaluations, the applicability of lifting surface theory based on potential flow as a prediction technique is limited.
8. Airplane stability characteristics are, in general, moderately affected by flexibility. The quasi-static-elastic formulation is usually adequate for most prediction tasks, dynamic effects being generally modest. However, some dynamic check cases should be run, since there is no assurance that dynamic effects will be small for any given configuration.

It is recommended that further development of prediction methods follow the plan\* mentioned above. Lifting surface theory and beam analysis should be used to determine aerodynamic and structural influence coefficients and to calculate stability derivatives and stability characteristics. For dynamic stability investigations, if the center-of-gravity disturbance is characterized by the small perturbation equations of motion, the characteristic equation rooting technique may be used. The stability of large perturbations is best evaluated by time history calculations.

\*Mechanization of this plan is currently under way under the direction of personnel of the Non-Steady Phenomena Branch, NASA-Ames.



## 2. INTRODUCTION

The stability of an airplane is its tendency to persist in a particular reference motion (for example, steady, level flight) when it has been disturbed from that motion. Primary factors affecting stability are the changes in the aerodynamic forces and moments acting on the airplane that occur with changes in the airplane's motion and orientation. They are expressed as stability derivatives evaluated at the reference motion condition, such as the change in airplane lift coefficient with change in angle of attack,  $\partial C_L / \partial \alpha|_{ref} \equiv C_{L\alpha}$ .

In the past, the effects of structural flexibility on airplane stability were accounted for by modifying the stability derivatives. For example, a change in airplane lift due to a change in angle of attack might produce twisting of the wing or bending of the fuselage, resulting in a different lift change for a particular angle-of-attack change than would occur had the airplane been rigid. The rigid airplane stability derivatives were then corrected to include these quasi-static-elastic effects.

The effects of structural dynamics have generally been a consideration only in flutter predictions. The center of gravity of the airplane was (and is) considered to be in a state of steady, level flight while the structure is disturbed. For example, neutral flutter stability is a constant-amplitude, oscillatory structural motion resulting from a disturbance; however, this motion is regarded as having no effect on the steady, level motion of the center of gravity of the airplane. This is a satisfactory representation of the motion provided the frequency of the structural motion is well separated from the natural frequencies of the overall motion of the airplane, e.g., its short period longitudinal natural frequency. In this case the two motions are not sufficiently coupled for an exchange of energy between them.

Certain aerodynamic and structural approximations have also been used. Aerodynamic surfaces—wings and tails—may have sufficiently large aspect ratios that lifting line theory may be used to predict aerodynamic loads and the structure may be treated essentially as an assemblage of beams. These and other approximations were satisfactory and led to relatively simple methods for predicting the influence of structural flexibility on airplane stability.

However, the advent of large airplanes operating in the transonic and supersonic flight regimes has led to configurations for which some of the acceptable approximations of the past are of questionable validity or are obviously invalid. The frequencies of the structural motion have been sufficiently reduced by the increase in both airplane flexibility and cruise dynamic pressure that coupling with overall motion of the airplane is attendant. The use of low aspect ratio aerodynamic surfaces invalidates the lifting line aerodynamic approximation and reduces the applicability of equivalent beam structural models. Different methods of analysis must be introduced.

Bisplinghoff and Ashley (ref. 1, chapter 9) have laid the ground work for new, less restrictive methods by presenting equations of motion that have the required degree of generality. Milne (ref. 2) also presents the development of more general equations of motion that integrate conventional stability and aeroelastic methods. Milne's rather extensive work also includes the application of the equations to the problem of slender airplane trim state and longitudinal stability. Most major airplane companies are also involved in developing new, less restrictive methods. However, results of these studies are usually not available in the open literature.

With this background in mind, an investigation was conducted for the purpose of developing an improved method for predicting the stability and control characteristics of an elastic airplane. Objectives of the study were to:

1. Develop equations of motion that have sufficient generality to handle large, flexible airplane stability and control problems, including dynamic-elastic effects.
2. List and evaluate assumptions and restrictions introduced and determine how they may influence the prediction of airplane stability and control characteristics.
3. Develop stability criteria applicable to flexible airplanes.
4. Develop improved methods for predicting elastic airplane stability derivatives using current and/or improved aerodynamic and structural techniques.
5. Develop an improved approach for predicting elastic airplane stability characteristics using the results of 1 through 4 above.
6. Document the results of the study in a precise, understandable form.

The work was accomplished under the technical direction of the Non-Steady Phenomena Branch, Ames Research Center, Moffett Field, California. Members of the Aerodynamics and Structures Staff of the Commercial Airplane Division of The Boeing Company at Renton, Washington conducted the investigation as a joint effort. Frequent coordination meetings and reviews with Ames representatives were held during the course of the contract.

The scope of the investigation included consideration of large, flexible airplanes operating in the flight envelope of fig. 1. Only "clean" configurations were studied; landing, takeoff, ground effects, stability augmentation, and control surface movements were not considered. Rectilinear and curvilinear reference flight paths were assumed, and arbitrary, large, and small airplane motion perturbations about the reference flight paths investigated. Structural motions of dynamic-elastic, quasi-static-elastic, and rigid nature were included in the study.

Only stability and control characteristics were included in the investigation; no consideration was given to the prediction of airplane performance, flutter, or structural loads. Both static and dynamic stability characteristics were studied. Stability criteria were also examined, but no handling-qualities work was included.

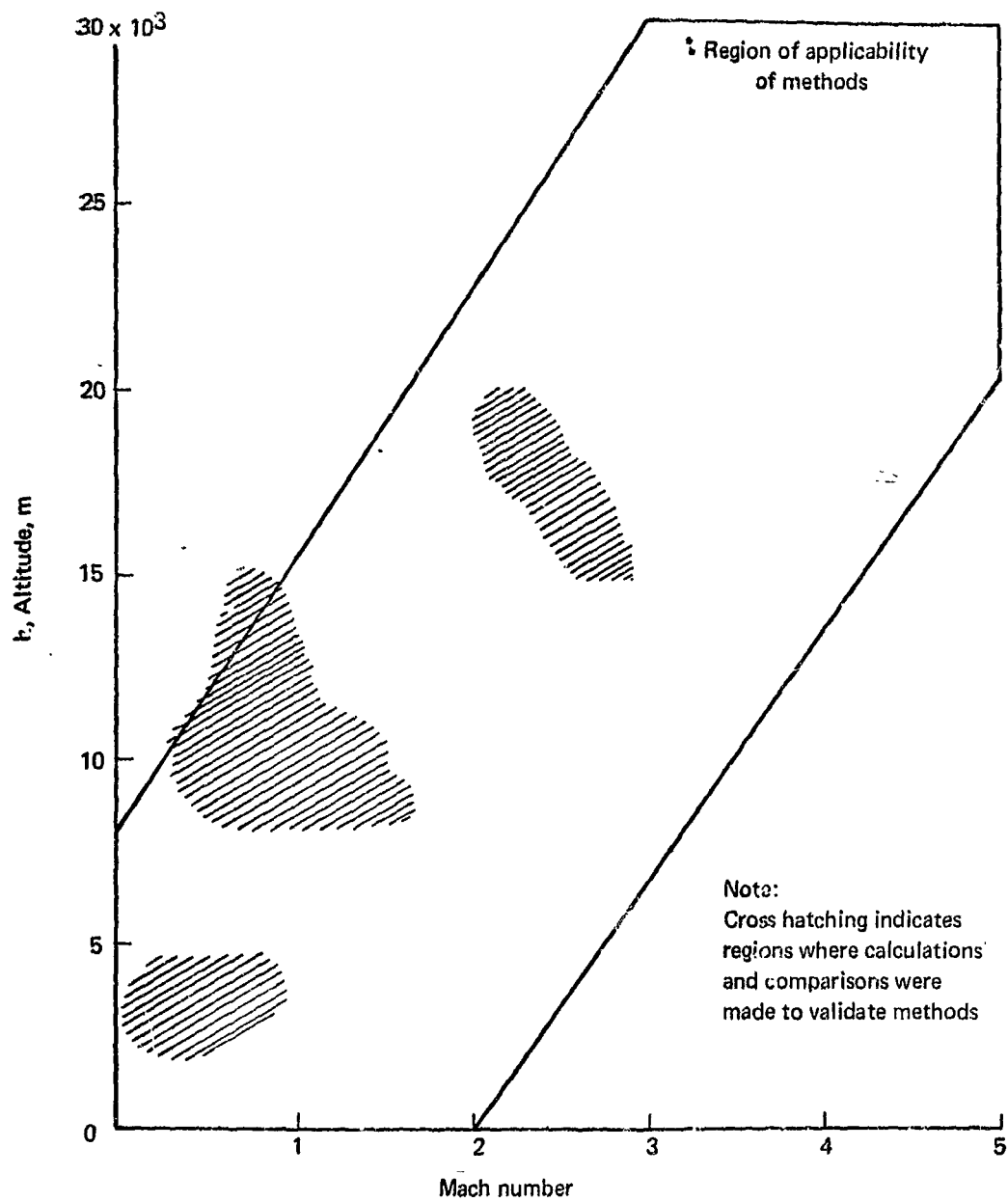


FIGURE 1. - FLIGHT ENVELOPE FOR INVESTIGATION

The investigation was further restricted to the consideration and development of analytical methods; experimental methods were of interest only to the extent that they served as a standard for comparing and evaluating the analytical approaches. Although the number of theoretical methods considered was large for the initial part of the investigation, it was rapidly narrowed for the major task of comparing and evaluating the several most promising approaches.

The approach taken in the investigation was to develop in order:

- equations of motion
- stability criteria
- stability derivative prediction methods
- stability characteristics prediction methods

Three mathematical models of an elastic airplane were considered; rigid (generally frozen in the reference shape), equivalent elastic\*, and dynamic elastic.

The rigid model admits no structural deflections from the shape in the reference motion. It serves as a base and provides a means for evaluating theoretical methods by comparing predictions with data obtained from essentially rigid wind tunnel models. The rigid model was sufficiently accurate to describe most aircraft prior to the introduction of large (weights of over 100 000 lb (45 359 kg)), swept-wing jet aircraft flying at dynamic pressures above 400 psf (19 152 N/m<sup>2</sup>).

The equivalent elastic model assumes that all structural deflections are of a quasi-static-elastic nature. Air and inertia loads are considered to be in phase with the deflections. No structural dynamic effects are included. This model has also been referred to as static-elastic or quasi-static-elastic in the literature. For this model, flight test and flexible-model wind tunnel data are used as the basis for validation. The equivalent elastic formulation is satisfactory wherever there is a reasonable frequency spread between structural and control modes.

The dynamic-elastic model is the most complex of the three. It takes into account dynamic motions of the structure as well as in-phase deflections. Usually this case is handled by considering 10 to 80 structural vibration modes. A variation of this approach that lends itself to computer calculation is the residual-flexibility formulation, which considers the correct phasing of the lower frequency vibrational modes but assumes that higher frequency modes are in phase with the loads. This approximation gives to the residual-flexibility method advantages for computer mechanization that may result in more accurate answers than would be obtained with the straightforward inclusion of modes with their

\*The terms quasi-static-elastic, static-elastic, and equivalent elastic are used interchangeably in this report.

mathematically correct phasing. The dynamic-elastic formulation is usually required for cases where the structural frequencies are near maneuver and control frequencies (approximately within a 2:1 ratio).

The equations of motion were developed first for the rigid airplane, treating first the general equations of motion, then airplane motion perturbations about the reference flight path. Next, the flexible airplane equations of motion were developed, using the lumped parameter concept in describing structural flexibility and structural equilibrium. Perturbation motions were united with the structural motions for both equivalent elastic and dynamic-elastic cases. Both residual-flexibility and completely elastic approaches were taken in developing the equations to describe the dynamic-elastic airplane case. Assumptions and restrictions introduced into the developments were carefully evaluated as to their effects on stability and control applications of the equations.

The approach taken in establishing stability criteria was essentially one of examining rigid airplane criteria to determine whether they do or do not apply to flexible airplanes. This part of the study was divided into consideration of static and dynamic stability cases. The usual mathematical approaches of characteristic equation rooting and time history traces to assess dynamic stability were taken. Some consideration was given to energy decay methods for possible future application.

Stability derivative prediction methods for rigid airplanes can involve the use of a variety of aerodynamic theories and methods. The approach taken here was to compare and evaluate against experimental data those methods based on lifting line theory; lifting surface theory; and handbook compilations of theoretical, empirical, and test data. For the elastic airplane, the structural methods considered were finite element theory and equivalent beam analysis. Structural methods were compared for the purpose of finding the one most suitable for preliminary design application.

Methods for the more extensive task of predicting the stability characteristics require a unification of the previously mentioned areas. Static stability characteristics were determined by using the stability derivatives. The dynamic stability calculation methods investigated consisted of approximate empirical handbook formulas, roots of the small perturbation characteristic equations, and time histories from solution of the large perturbation equations of motion.

The approach taken for this part of the study was to investigate the regions of applicability of the available methods and to assess the relative importance of static-elastic and dynamic-elastic effects on airplane stability. Areas in the flight envelope of fig. 1 where specific calculations and comparisons were made to validate the methods are shown by cross hatching. Application of the methods to other areas within the envelope was justified by an evaluation of the assumptions and restrictions incorporated into the governing equations.

The report consists of four separately bound volumes: a summary report and three appendixes. These also serve as a handbook which describes and discusses the pertinent aeroelastic methods.

The summary report presents the results and conclusions of the study with discussions as required for the reader to gain an understanding of the subject. It should be useful for managers, those new to the field or experienced only in related fields, and any others desiring an overview of the subject.

The appendixes contain those results of interest primarily to the specialist, including the detailed steps and discussion of the various derivations and developments as well as the variations in approaches and applications of the methods. Appendix A treats the development of the equations of motion and stability criteria. Appendix B develops and evaluates methods for determining longitudinal and lateral-directional stability derivatives. Appendix C, using the results of the previous work, discusses and evaluates the pertinent factors important to the prediction of airplane stability and response characteristics.

Matrix notation and methods have been used in developing and presenting many of the equations in the summary report and succeeding parts of the study. The reader not familiar with the notation is advised to spend the short time necessary to learn the basic ideas and symbols of matrix methods. For the specialist, a thorough understanding is essential. Reference 3, or one of the many other available books on matrix methods, is recommended.

It is also assumed that the reader has some background in the field of stability and control. If this is not the case, or if review is desired, Etkin's book (ref. 4) is suggested for study.

### 3. SYMBOLS

This list includes the symbols found in the Summary and appendixes. In different technologies some of the symbols have different meanings. For example,  $\epsilon$  means downwash angle to an aerodynamicist, but strain to a structural engineer. In these cases the several definitions have been listed after the symbol.

#### General

$A$	Aspect ratio, nondimensional
$[A]$	Steady aerodynamic influence coefficients matrix, meters <sup>2</sup> /radian
$[\delta A]$	Unsteady aerodynamic influence coefficients matrix, meter <sup>2</sup> -seconds/radian
$[A_1], [A_2], [A_3], [A_4], [A_5]$	Aerodynamic matrices, newtons, newton-meters
$a$	Root of characteristic equation, second <sup>-1</sup> ; lift curve slope, radian <sup>-1</sup>
$a_\infty$	Speed of sound, meters/second
$\bar{a}_v$	Vertical tail elastic to rigid lift ratio, nondimensional
$\bar{a}$	Acceleration, meters/second <sup>2</sup>
$b$	Wingspan, meters
$C_{1/2}$	Cycles to damp to half amplitude, nondimensional
$C_2$	Cycles to double amplitude, nondimensional
$C_D$	Drag coefficient, $D / \bar{q}S$ , nondimensional
$C_{D_i}$	Induced drag coefficient, $D_i / \bar{q}S$ , nondimensional
$C_L$	Lift coefficient, $L / \bar{q}S$ , nondimensional
$C_l$	Rolling moment coefficient, $M_x / \bar{q}Sb$ , nondimensional

$C_m$	Pitching moment coefficient, $M_y / \bar{q} S \bar{c}$ , nondimensional
$C_N$	Normal pressure force coefficient, $N / \bar{q} S$ , nondimensional
$C_n$	Yawing moment coefficient, $M_z / \bar{q} S b$ , nondimensional
$C_p$	Pressure coefficient, $(P - P_\infty) / \bar{q}_\infty$ , nondimensional
$C_T$	Thrust coefficient, $T / \bar{q} S$ , nondimensional
$C_Y, C_y$	Side force coefficient, $F_y / \bar{q} S$ , nondimensional
$\{C\}$	Flexibility matrix with reference point fixed, meters/newton
$\{C_o\}$	Flexibility matrix with reference point fixed and with reference point rows and columns removed, meters/newton
$\{\bar{C}\}$	Flexibility matrix with reference point free, meters/newton
$\{\bar{C}_R\}$	Residual flexibility matrix, meters/newton
$c$	Wing chord, meters
$c_R$	Root chord, meters
$\bar{c}$	Mean aerodynamic chord, meters
$c_{ref}$	$\bar{c}$ for the 707 and $c_R$ for the SST, meters
$D$	Drag, newtons
$D_i$	Induced drag, newtons
$\{D\}$	Transformation matrix from fluid to stability axis system, nondimensional
$\bar{d}$	Elastic displacement, meters
$\{d_i\}$	Column matrix of elastic displacement components at the $i^{th}$ element, meters
$\{d_p\}$	Matrix of elastic displacement perturbation, meters
$E$	Total airplane perturbation energy, newton-meters; Young's modulus, newtons/meter <sup>2</sup> ; induced drag efficiency factor, nondimensional; energy, newton-meters



$e$	Internal energy density, newton-meters <sup>4</sup> /kilogram
$F$	Energy decay parameter, nondimensional
$\vec{F}$	Force, newtons; surface stress vector, newtons/meter <sup>2</sup>
$\{F\}$	Total force matrix, newtons
$\{F_A\}$	Aerodynamic force matrix, newtons
$[F_d]$	Flexibility matrix relating changes in panel centroid deflections to unit loads, meters/newton
$\{F_i\}$	Generalized forces at $i^{\text{th}}$ element, arbitrary dimensions
$\{F_T\}$	Thrust force matrix, newtons
$[F_\Theta]$	Flexibility matrix relating panel slopes to unit loads, radians/newton
$f_{ij}$	Aerodynamic influence coefficients (subsonic), newtons/radian
$\vec{f}$	Perturbation force, newtons; perturbation surface stress vector, newtons/meter <sup>2</sup>
$\{f\}$	Perturbation force matrix, newtons
$\{f_A\}$	Aerodynamic perturbation force matrix, newtons
$\{f_T\}$	Thrust perturbation force matrix, newtons
$G$	Shear modulus, newtons/meter <sup>2</sup>
$GW$	Gross weight, newtons
$\bar{G}$	Structural influence functions in diadic form with reference point free, meters <sup>3</sup> /newton
$g_{ij}$	Aerodynamic influence coefficients (supersonic), newtons/radian
$\vec{g}$	Acceleration due to gravity, meters/second <sup>2</sup>
$\vec{g}_i$	Unit base vector, nondimensional
$h$	Altitude, meters; specific enthalpy, newton-meters/kilogram; center-of-gravity position, nondimensional

$h_m$	Maneuver point position, nondimensional
$h_n$	Neutral point position, nondimensional
$(h_n - h)$	Static margin, nondimensional
$\dot{h}_p$	Velocity of panel normal to the streamwise direction, meters/second
$I_{xx}, I_{xy}, I_{xz}$ $I_{yy}, I_{yz}, I_{zz}$	Moments and products of inertia, kilogram-meters <sup>2</sup>
$[I], [\bar{I}]$	Identity matrix, nondimensional
$i_H$	Horizontal tail deflection, degrees
$\hat{i}, \hat{j}, \hat{k}$ $\underline{i}, \underline{j}, \underline{k}$	Unit base vectors, nondimensional
$J$	Torsional constant, meters <sup>4</sup> /radian
$K$	Angular deflection at the exposed horizontal tail due to a unit load at the tail, radians/newton
$K_{ij}$	Structural stiffness coefficient, newtons/meter
$K_N$	Ratio of aircraft nose lift to aircraft wing lift, nondimensional
$K_{\dot{p}}$	Effective change in vertical tail angle of sideslip due to a unit change in rolling acceleration measured at the exposed vertical tail, degrees/radian/second <sup>2</sup>
$K_{\dot{r}}$	Effective change in vertical tail angle of sideslip due to a unit change in yawing acceleration measured at the exposed vertical tail, degrees/radian/second <sup>2</sup>
$K_{\dot{y}}$	Effective change in vertical tail angle of sideslip due to a unit change in side acceleration measured at the exposed vertical tail, degrees/meter/second <sup>2</sup>
$K'_{B(W)}$	Effect of lift carryover on the body due to the wing, nondimensional
$K'_{W(B)}$	Effect of lift carryover on the wing due to the body, nondimensional
$\{K\}$	Stiffness matrix with respect to fixed reference point, newtons/meter

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$\{K\}_i$	Element stiffness matrix, newtons/meter
$\{\bar{K}\}$	Stiffness matrix with respect to free reference point, newtons/meter
$[\bar{K}]$	Generalized stiffness matrix with free reference point, newtons/meter
$k$	Thermal conductivity, newton-meters/second-meter-degrees Celsius; elastic constant, newtons/meter <sup>2</sup> ; Strouhal number, nondimensional
$\{K\}, [K]$	Corrector matrix for influence coefficients, nondimensional
$L$	Lift, newtons
$l$	Moment arm, meters; characteristic length, meters; pressure difference across surface, newtons/meter <sup>2</sup>
$l_H$	Wing $c_{ref}/4$ to horizontal tail $c_{ref}/4$ , meters
$l_V$	Wing $c_{ref}/4$ to vertical tail $c_{ref}/4$ , meters
$l_1, l_2, l_3$	Direction cosines, nondimensional
$M$	Mach number, nondimensional; mass of the airplane, kilograms
$\bar{M}$	Moment, meter-newtons
$\{M\}$	Inertial matrix, kilograms, kilogram-meters <sup>2</sup>
$[M]$	Generalized mass matrix, kilograms
$m_1, m_2, m_3$	Direction cosines, nondimensional
$\bar{m}$	Perturbation moment, meter-newtons
$\{m\}$	Mass matrix, kilograms
$[m]$	Diagonal mass matrix, kilograms
$N$	Yawing moment, meter-newtons
$\bar{N}$	Normal force, newtons
$n$	Load factor, nondimensional; number of elastically connected mass elements used to represent the airplane, nondimensional

$n_1, n_2, n_3$	Direction cosines of the normal surface, nondimensional
$\hat{n}$	Unit vector normal to the surface, nondimensional
$[n]$	Diagonal matrix of panel unit normal vectors, nondimensional
$P$	Period, seconds
$P, Q, R$	Components of the angular velocity $\vec{\omega}$ in the body axis system, radians/second
$P_t$	Total pressure, newtons/meter <sup>2</sup>
$\{P\}$	Aerodynamic panel pressure forces, newtons
$p$	Static pressure, newtons/meter <sup>2</sup> ; roll rate, radians/second
$p, q, r$	Perturbation components of angular velocity $\vec{\omega}_p$ in the body axis system, radians/second
$Q_i$	Generalized force, arbitrary dimensions*
$\{Q\}$	Matrix of generalized aerodynamic and thrust forces, arbitrary dimensions*
$q$	Pitch rate, radians/second; rate of internal heat energy addition, newton-meters/second
$q_i$	Generalized coordinates, arbitrary dimensions*
$\bar{q}$	Dynamic pressure, newtons/meter <sup>2</sup>
$\hat{q}$	Pitch rate, $q c_{ref}/2 V_{c1}$ , nondimensional
$\{q\}$	Matrix of generalized coordinates, arbitrary dimensions*
$\{\tilde{q}\}$	Matrix of generalized coordinates of elastic free vibration, arbitrary dimensions*
$\{\tilde{\tilde{q}}\}$	Cantilever eigenvectors, nondimensional

\*The units of a generalized force times the generalized coordinates must be newton-meters.

$R$	Universal gas constant, newton-meters/kilogram-degrees Kelvin; magnitude of position vector, meters; region of XY plane not covered by the airplane or wake, nondimensional
$Re$	Reynolds number, nondimensional
$\vec{R}$	Position vector at an initial instant of time, meters; body force per unit volume, newtons/meter <sup>3</sup>
$r$	Reference distance, meters; magnitude of the position vector, meters
$\hat{r}$	Yaw rate component, $rb/2V_{c1}$ , nondimensional
$\vec{r}$	Position vector relative to the body axis system, meters; position vector relative to the fluid axis system, meters
$\vec{r}_O$	Position vector of the center of gravity relative to the fluid axis system, meters
$\vec{r}_s$	Position vector relative to the stability axis system, meters
$\vec{r}^*$	Position vector relative to inertial space, meters
$\vec{r}_O^*$	Position vector of the center of gravity relative to the inertial space, meters
$\vec{r}_s^*$	Position vector in the undeformed airplane relative to the body axis system, meters
$\{r'_{Op}\}$	Matrix of airplane position and orientation perturbations, meters, radians
$S$	Reference area, meters <sup>2</sup> ; airplane's projection on the XY plane, nondimensional
$[S]$	Diagonal matrix of panel areas, meters <sup>2</sup>
$s$	Complex frequency function, 1/seconds
$T$	Kinetic energy, newton-meters; thrust, newtons; time, seconds
$T_{1/2}$	Time to damp to 1/2 amplitude, seconds
$T_2$	Time to double the amplitude, seconds

$-1/T_r$	Rolling convergence mode root, 1/seconds
$-1/T_s$	Spiral mode root, 1/seconds
$t$	Time, seconds; airfoil thickness, meters
$t^*$	Nondimensionalizing time factor, seconds
$U$	Potential energy, newton-meters
$U, V, W$	Components of velocity $\vec{V}_c$ in the body axis system, meters/second
$u, v, w$	Perturbation components of the velocity in the body axis system, meters/second
$u_i$	Generalized coordinates, nondimensional
$\hat{u}$	Forward velocity component, $u/V_{c1}$ , nondimensional
$\{u\}, \{u_p\}$	Generalized elastic displacements, meters
$V$	Lyapunov function, nondimensional; volume, meters <sup>3</sup>
$V_E$	Equivalent airspeed, meters/second
$\vec{V}_c$	Velocity vector of the airplane center of gravity, meters/second
$\vec{V}$	Velocity vector, meters/second
$\vec{V}_{cp}$	Perturbation velocity vector of the airplane center of gravity meters/second
$\{V_p\}$	Matrix of airplane linear and rotational rate perturbations, meters/second, radians/second
$\{\dot{V}_p\}$	Matrix of airplane linear and rotational acceleration perturbations, meters/second <sup>2</sup> , radians/second <sup>2</sup>
$W$	Weight, newtons; airplane's wake projection on the XY plane, nondimensional
$\{X\}$	Matrix of panel centroid distances to the reference point, meters
$X, Y, Z$ $x, y, z$	Body-fixed-axis system (app. A); fluid axis system (app. B) Perturbation components of the velocity in the body axis system

$X_B, Y_B, Z_B;$ $x_B, y_B, z_B$	Body-fixed-axis system
$X_O, Y_O, Z_O$	Axis system fixed to a material point
$X', Y', Z';$ $x', y', z'$	Earth-fixed-axis system
$Y$	Side force, newtons
$\uparrow \Delta y \downarrow$	Matrix of spanwise panel widths, meters
$Z_R$	Vertical displacement of structural reference point, meters
$\{Z\}$	Matrix of vertical displacements of each panel from equilibrium, meters
$[ \ ]$	Square matrix
$\{ \}$	Column matrix
$( \ )$	Row matrix
$\uparrow \downarrow$	Diagonal matrix
$[ \ ]^T, \{ \}^T$	Transposed matrix
$[ \ ]^{-1}$	Matrix inverse
$  [ \ ]  $	Determinant of a matrix
$[0]$	All zero elements
$\{1\}$	Column matrix of ones
$\ll \gg$	“Jump” in enclosed quantity
Greek Symbols	
$\alpha$	Angle of attack, radians
$\alpha_R$	Angular rotation of structural reference point, radians
$\alpha_{ref}$	Angle between $X$ body axis and $\vec{V}_{c_1}$ , radians
$\{\alpha\}$	Matrix of panel slopes, radians

$\beta$	Angle of sideslip, radians
$\beta^2$	$(M^2 - 1)$ , nondimensional
$\Gamma$	Circulation, meters <sup>2</sup> /second
$\bar{T}_0$	Structural influence functions with reference point fixed in diadic form, meters <sup>3</sup> /newton
$\gamma$	Flight path angle, radians; ratio of specific heats for air, nondimensional
$\Delta$	Finite change in some parameter, nondimensional
$\delta$	Control surface deflection, radians; arbitrarily small number, nondimensional; Dirac's function, nondimensional; thickness ratio, nondimensional
$\{\delta\}$	Matrix of displacements relative to a space-fixed inertial system, meters
$\{\delta_s\}$	Matrix of flexible displacements relative to the structural axis system, meters
$\epsilon$	Downwash angle, radians; arbitrarily small number, nondimensional; strain, meters/meter
$\epsilon_\alpha$	Change in downwash angle at the stabilizer per unit change in wing angle of attack, $\partial\epsilon/\partial\alpha$ , radians/radian
$\zeta$	Damping ratio, nondimensional; nondimensionalized coordinate, nondimensional; dummy variable, nondimensional
$\eta$	Efficiency factor, nondimensional; coordinate, nondimensional; dummy variable, nondimensional
$\Theta$	Euler angle, radians
$\theta$	Perturbed Euler angle, radians
$\Theta_s$	Streamwise rotation of panel, radians
$\Theta_{ix}, \Theta_{iy}, \Theta_{iz}$	Node rotations, radians
$\dot{\Theta}$	Rate of change of Euler angle, radians/second



$\dot{\theta}_{ei}$	Rotational rate of paneled airplane about axis of rotation, radians/second
$\bar{\theta}$	Rigid-body rotation about center of gravity, radians
$[\theta]$	Angle mode matrix, radians/meter
$\lambda$	Eigenvalue, nondimensional; taper ratio, nondimensional; bulk modulus, newtons/meter <sup>2</sup> ; Lamé's constant, newtons/meter <sup>2</sup> ; sweep angle, degrees
$\lambda_i$	Roots of characteristic equation, 1/seconds
$\mu$	Reduced mass parameter, nondimensional; Lamé's constant, newtons/meter <sup>2</sup> ; extent of influence region, nondimensional
$\{\mu\}$	Cantilever mode shape matrix, nondimensional
$[\mu]$	Matrix of all cantilever modes, nondimensional
$\nu$	Poisson's ratio, nondimensional
$\xi, \eta, \zeta$	Coordinates, nondimensional; dummy variables, nondimensional
$\pi$	Constant, 3.14159. . ., nondimensional
$\rho$	Density, kilograms/meter <sup>3</sup>
$\sigma$	Normal stress, newtons/meter <sup>2</sup> ; density ratio, nondimensional; real root of characteristic equation, 1/seconds
$\sigma_R$	Rotation of structural reference axis system, radians
$\sigma_T$	Rectilinear translation of structural reference axis system, meters
$\tau$	Coefficient of viscosity, kilograms/meter-second; shear stress, newtons/meter <sup>2</sup> ; time, nondimensional
$\Phi$	Total velocity potential, meters <sup>2</sup> /second; Euler angle, radians
$[\Phi_n]$	Normalized natural free vibration modes of the airplane, nondimensional

$\phi$	Perturbation velocity potential, meters; perturbed Euler angle radians
$\dot{\phi}$	Rate of change of Euler angle, radians/second
$[\phi]$	Free-vibration mode shape matrix, nondimensional
$[\bar{\phi}]$	Rigid-body mode shape matrix, nondimensional
$\bar{\phi}$	Stress diadic, newtons/meter <sup>2</sup>
$\bar{\phi}_\alpha$	Normal mode of generalized coordinate, nondimensional
$\varphi$	Velocity potential, nondimensional
$\varphi(t)$	Arbitrary positive function of time, arbitrary dimension
$\Psi$	Euler angle, radians
$\psi$	Perturbed Euler angle, radians
$\dot{\psi}$	Rate of change of Euler angle, radians/second
$\bar{\psi}$	Inertia diadic
$\Omega$	Phase angle, radians
$\omega$	Frequency, radians/second; imaginary part of a pair of complex roots, 1/seconds
$\omega_n$	Undamped natural frequency, radians/second
$\bar{\omega}_p$	Perturbed angular velocity, radians/second

#### Subscripts

A	Aerodynamic; airplane; aileron
a	Aerodynamic
ac	Aerodynamic center
b	Body reference axis system

<b>cg</b>	<b>Center of gravity</b>
<b>cp</b>	<b>Center of pressure</b>
<b>D</b>	<b>Dutch roll mode</b>
<b>E</b>	<b>Equivalent elastic (Formulation II); elevator</b>
<b><math>\bar{E}</math></b>	<b>Equivalent elastic (Formulation I)</b>
<b>Eff</b>	<b>Effective</b>
<b>EqEl</b>	<b>Equivalent elastic</b>
<b>exp</b>	<b>Experimental</b>
<b>F</b>	<b>Flutter</b>
<b>HB</b>	<b>Handbook methods</b>
<b>h, ht</b>	<b>Horizontal tail</b>
<b>I</b>	<b>Inertia relief</b>
<b><math>l</math></b>	<b>Lower surface</b>
<b>L.E. , LE</b>	<b>Leading edge</b>
<b>ls</b>	<b>Lifting surface theory method</b>
<b>P</b>	<b>Phugoid mode</b>
<b>R</b>	<b>Rigid; rudder</b>
<b>r</b>	<b>Rolling convergence root mode</b>
<b>S</b>	<b>Spiral root</b>
<b>sp</b>	<b>Short period</b>
<b>s</b>	<b>Stability axis system; spiral mode</b>

<b>sl</b>	<b>Sea level</b>
<b>t</b>	<b>Tip; total</b>
<b>u</b>	<b>Upper surface</b>
<b>v, vert, V.T.</b>	<b>Vertical tail</b>
<b>W</b>	<b>Wing</b>
<b>WB</b>	<b>Wing-body</b>
<b>WBT</b>	<b>Wing-body-tail</b>
<b>WT</b>	<b>Wind tunnel</b>
<b>0</b>	<b>At <math>\alpha = \delta_E = i_h = 0^0</math>; initial state</b>
<b>1</b>	<b>Steady state motion variables; trimmed condition</b>
<b><math>\infty</math></b>	<b>Undisturbed condition</b>

#### **4. ASSUMPTIONS**

Assumptions used in developing the equations and methods are listed here for reference. Where appropriate in the summary report, pertinent assumptions used in obtaining a result or equation are given. However, discussions of the assumptions as they come into the developments are given in the appendixes. Further descriptions and justifications are included in those discussions.

### General Assumptions

- (G1) Airplane mass and mass distribution are constant with time
- (G2) No thermoelastic effects considered
- (G3) No electromagnetic effects considered
- (G4) Symmetric airplane
- (G5) Variation of air density with altitude is negligible
- (G6) No gust effects considered
- (G7) Gravitational forces on the field are negligible
- (G8) Small perturbation theory
- (G9) Large perturbation theory
- (G10) Origin of coordinate system is at the center of mass
- (G11) Arbitrary perturbations

### Aerodynamic Assumptions

- (A1) Potential flow theory
- (A2) Thin body
- (A3) Slender body
- (A4) High aspect ratio
- (A5) Prandtl boundary layer approximation
- (A6) Perfect gas, thermally nonconducting and chemically nonreacting
- (A7) Isentropic flow
- (A8) Steady flow

- (A9) Unsteady flow
- (A10) Inviscid flow
- (A11) Quasi-steady flow
- (A12) Aerodynamic influence coefficients for nonzero sideslip
- (A13) Continuum flow
- (A14) No finite shock waves
- (A15) Velocity field is irrotational

#### Structural Assumptions

- (S1) Hooke's law applies
- (S2) Only small strain and displacement gradients are considered
- (S3) Structural damping is negligible
- (S4) Structural perturbations can be represented by normal modes
- (S5) Completely elastic math model of elastic airplane
- (S6) Residual elastic math model of elastic airplane
- (S7) Equivalent elastic math model of elastic airplane
- (S8) Rigid math model of elastic airplane
- (S9) Airplane displacement vector field is such that the center of gravity does not displace or rotate
- (S10) X component of elastic deflection is negligible
- (S11) Y component of elastic deflection is negligible
- (S12) The structure can be adequately represented with beams
- (S13) Inertia of each finite mass element about its center of gravity is negligible

## Dynamic Assumptions

- (D1) Free flight only
- (D2) No spinning rotors
- (D3) Steady-state curvilinear flight
- (D4) Steady-state rotation is small
- (D5) Zero-lag thrust derivatives
- (D6)  $C_{L\ddot{\theta}}$  is negligible
- (D7)  $C_{Y\dot{p}_I}$ ,  $C_{Y\dot{r}_I}$ ,  $C_{L\ddot{Y}_I}$ , and  $C_{n\ddot{Y}_I}$  are negligible
- (D8)  $C_{Dq}$  is negligible
- (D9) Steady-state rectilinear motion
- (D10) Stick-fixed-and-unaugmented airplane
- (D11) Thrust perturbation forces are negligible
- (D12) Steady state, wings level, and zero sideslip
- (D13) Level flight (steady state)
- (D14) Linear aerodynamic stability derivatives
- (D15) Two-degree-of-freedom longitudinal motion



## 5. EQUATIONS OF MOTION

The equations of motion for rigid airplanes and flexible airplanes are presented in this section in the forms deemed most suitable for assessing airplane stability characteristics. The stability of an airplane is its tendency to persist in a steady reference motion when it has been disturbed from that reference motion. The steady reference motion must be defined and the perturbation motion analyzed. Thus, four sets of equations of motion are presented. Two govern the reference and perturbation motions for rigid airplanes and two govern the reference and perturbation motions for flexible airplanes.

The perturbation equations of motion are presented for three different order-of-magnitude approximations of the size of the perturbation motion variables. These are termed arbitrary, large, and small perturbation equations of motion. The small perturbation equations are those which have had greatest application in the analysis of the stability of a steady reference motion. Arbitrary and large perturbation equations may be used in investigations involving large disturbances that upset the airplane and in the study of maneuvering flight. However, since neither upset nor maneuvering flight are within the scope of this study, the small perturbation equations of motion are given primary emphasis.

The equations of motion presented for rigid airplanes are those which are familiar to engineers working airplane stability and control problems. They are essentially those developed by Etkin (ref. 4); however, here they are expressed in a body-fixed-axis system that is convenient for the application of aerodynamic influence coefficient methods. This is in contrast with the usual formulation in the stability axis system (ref. 4). In addition, the formulation provides for very general steady-reference motions including curvilinear flight.

There is considerable diversity in the manner in which the equations of motion for flexible airplanes have been formulated. Modifications have been made in the past to accommodate new methods for predicting the aerodynamic forces on the airplane. This is apparent from a review of works on this subject such as those by Bisplinghoff and Ashley (ref. 1), and Milne (ref. 2).

The formulation used here is called a lumped parameter formulation. It facilitates the use of aerodynamic influence coefficients that relate a change in aerodynamic force on a small region of the airplane's surface, a panel, to an average change in flow incidence at another panel. This aerodynamic representation is particularly well suited for making empirical corrections to account for separated flow, viscous phenomena, and other aerodynamic phenomena that are not readily predicted theoretically but are of considerable concern to the stability and control engineer.

The basis for the lumped parameter formulation is described briefly in this section and in more detail in app. A. A complete description may be found in Bisplinghoff and Ashley (ref. 1). It usually is used in flutter analysis, although it has also been used to predict the longitudinal stability of flexible airplanes. It is the basis for the prediction of the longitudinal stability characteristics of the SST configuration appearing in this volume and in app. C. The formulation appearing in this section and in app. A is more general. This extended generality is included in the expectation that an aerodynamic influence coefficient method for wing-body combinations in nonsymmetrical motion can be developed. It also incorporates the useful approximation called residual flexibility, which is introduced by Schwendler and MacNeal (ref. 5).

The presentation appears in the form of a derivation. This form of presentation is used merely to introduce the principles on which the equations are based and to delineate the most important approximations which are included in their formulation. The derivation is not complete, although it may appear extensive in the flexible airplane case because of its inherent complexity. A detailed derivation appears in app. A; related derivations concerning aerodynamic and structural theories are provided in app. B.

Figure 2 summarizes the various forms for the equations of motion appearing in this section and gives a brief description of the approximations that characterize them.

## 5.1 Equations of Motion For a Rigid Airplane

5.1.1 General equations of motion.— The equations that govern the motion for a rigid airplane follow directly from fundamental principles of mechanics. These are the laws of conservation of momentum, which state that the rate of change of linear momentum is equal to the total force applied to the airplane and that the rate of change of angular momentum about the center of gravity is equal to the total applied force couple (or moment) about the center of gravity. They are stated analytically as

G1

$$M \frac{d\vec{V}_c}{dt} = M \vec{g} + \int_S \vec{F} dS \quad (1)$$

G2

G3

$$\frac{d}{dt} (\vec{\psi} \cdot \vec{\omega}) = \int_S \vec{r} \times \vec{F} dS \quad (2)$$

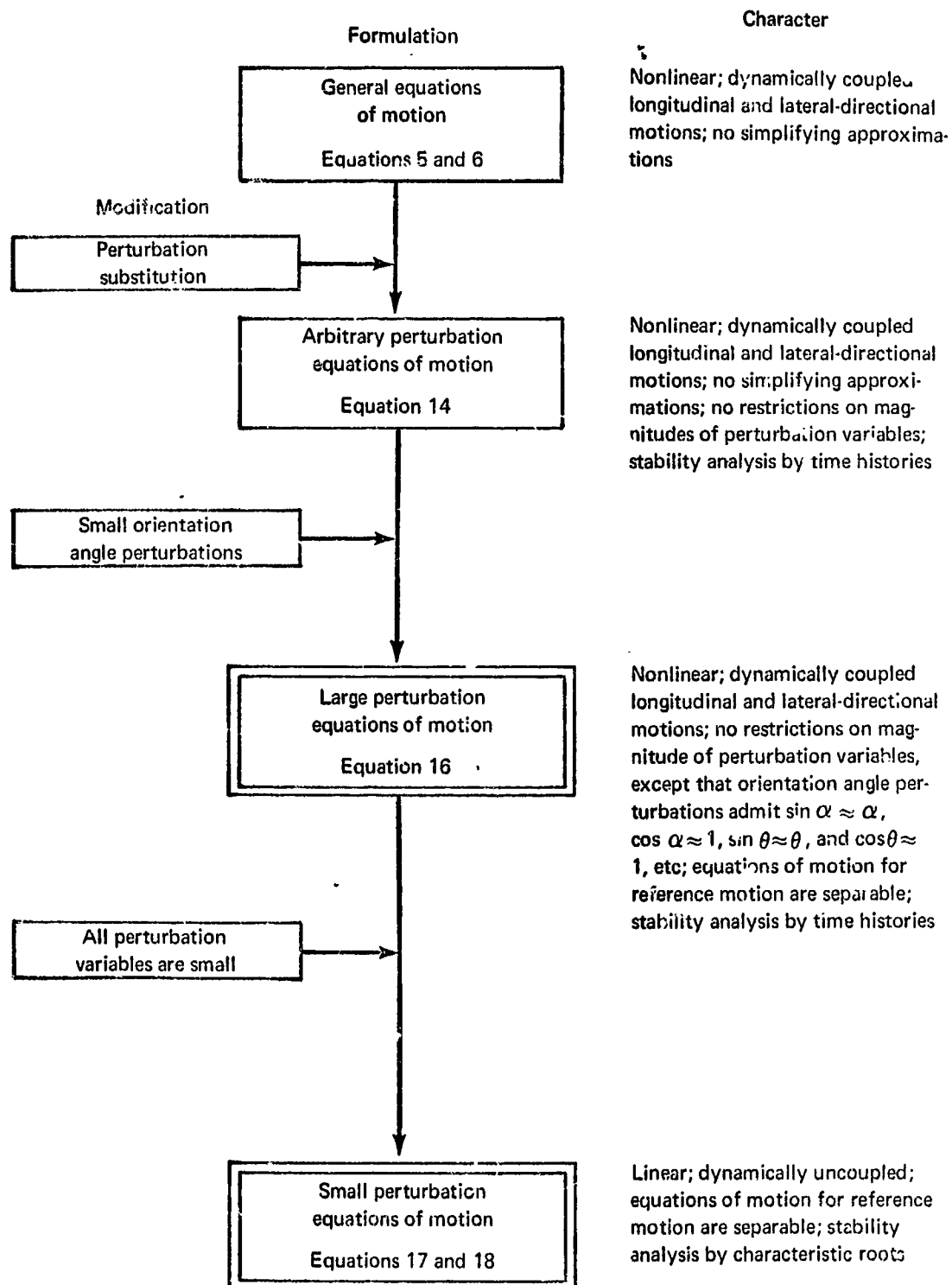


FIGURE 2.— MODIFIED FORMS OF EQUATIONS OF MOTION

where

$M$  = total airplane mass

$\vec{V}_c$  = velocity of the airplane c.g. relative to earth-fixed-axis system  $(x', y', z')$ , fig. 3

$\vec{g}$  = gravity force per unit mass

$\vec{F}$  = surface force per unit area

$S$  = airplane's total surface

$\psi$  = inertia tensor expressed as a diadic

$= \vec{i} I_x \vec{i} + \vec{j} I_y \vec{j} + \vec{k} I_z \vec{k} - \vec{i} I_{xz} \vec{k} - \vec{k} I_{zx} \vec{i}$

$\vec{\omega}$  = rate of rotation of airplane relative to earth-fixed-axis system

$\vec{r}$  = position of an arbitrary  $Q$  point relative to airplane c.g., fig. 3

(G10)

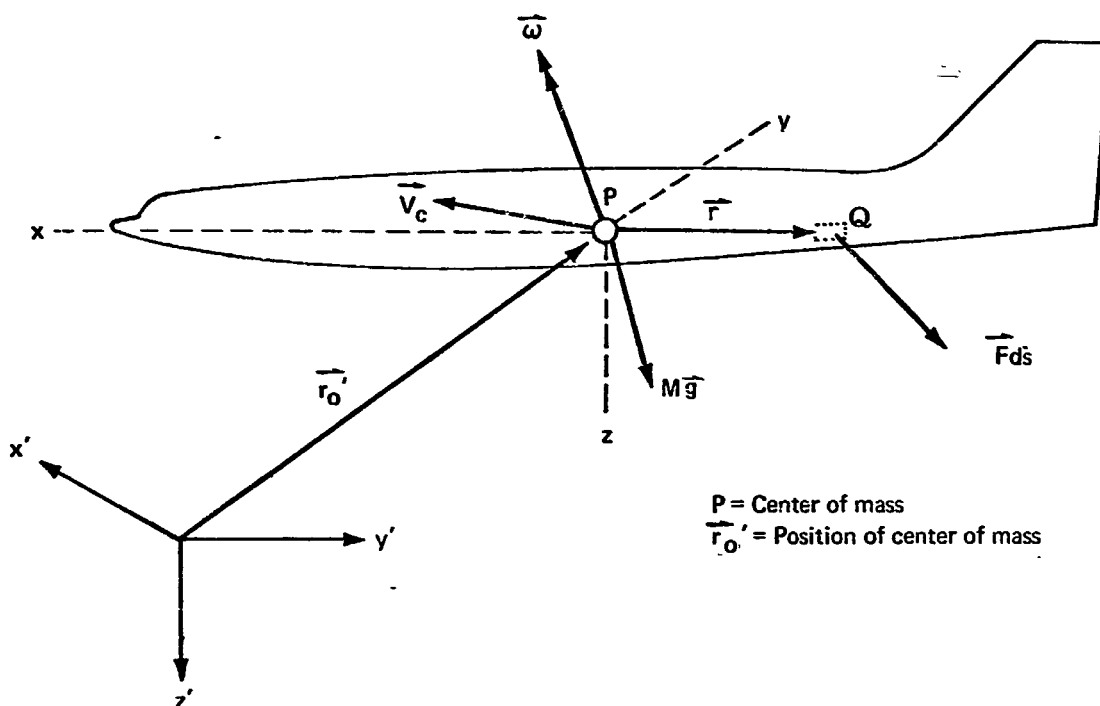


FIGURE 3.— AXIS SYSTEM FOR THE UNDEFORMED SHAPE OF AN ELASTIC AIRPLANE

The time rate of change  $d/dt$  is that apparent to an observer in the earth-fixed-axis system  $x', y', z'$ , which is assumed to be an inertial reference frame. Thus, letting  $\partial/\partial t$  represent the time rate of change apparent to an observer in a body-fixed-axis system  $x, y, z$  and letting a dot ( $\dot{\phantom{x}}$ ) represent the partial derivative  $\partial/\partial t$ , the equation of motion may be expanded as

$$M (\vec{\dot{V}}_c + \vec{\omega} \times \vec{V}_c) = M\vec{g} + \vec{F}_A + \vec{F}_T \quad (3)$$

$$\vec{\psi} \cdot \vec{\dot{\omega}} + \vec{\omega} \times (\vec{\psi} \cdot \vec{\omega}) = \vec{M}_A + \vec{M}_T \quad (4)$$

where

$$\vec{F}_A + \vec{F}_T = \int_S \vec{F} dS$$

and

$$\vec{M}_A + \vec{M}_T = \int_S \vec{r} \times \vec{F} dS$$

and where  $\vec{F}_A$  and  $\vec{M}_A$  are the total force and moment due to aerodynamics and  $\vec{F}_T$  and  $\vec{M}_T$  are the total force and moment due to the thrust of the propulsion units.

Equations (3) and (4) represent the general equations of motion for a rigid airplane. The x,z plane of the body axis system has been made to coincide with the plane of symmetry of the airplane so that the inertia diadic contains only the  $I_{xz}$  product of inertia. Certain assumptions have been used in writing these equations. They are:

Free flight only

Origin of coordinate system at the center of gravity

Airplane mass and mass distribution constant with time

No consideration of electromagnetic effects

No spinning rotors

Symmetric airplane

(D1)

(G10)

(G1)

(G3)

(D2)

(G4)

Equations (3) and (4) may be expanded as six scalar equations, but before they may be so expanded the gravitation force vector must be written in terms of the body axis system. A convenient form is obtained in terms of Euler orientation angles. These angles orient the airplane in space relative to the earth-fixed-axis system, as shown by fig. 4. Let  $z'$  in the earth-fixed system be directed positively down toward the earth's center, and let  $x'$  have some specified direction. Let axis system  $x_1, y_1, z_1$  be initially codirectional with the  $x', y', z'$  system but with origin at the airplane's c.g. Introduce the rotation  $\psi$  about the  $z_1$  axis, which rotates a system  $x_2, y_2, z_2$  such that  $x_2$  is in the plane of airplane symmetry. Then rotate a  $x_3, y_3, z_3$  system about  $y_2$  through the angle  $\theta$  so that  $x_3$  coincides with  $x$ . Finally, rotate about  $x_3$  through the angle  $\phi$  so that  $y_3$  coincides with  $y$ . This completely orients the airplane relative to the earth-fixed-axis system. One may choose alternate definitions of the Euler angles; the definition presented here is the one used by Etkin (ref. 4, p. 100).

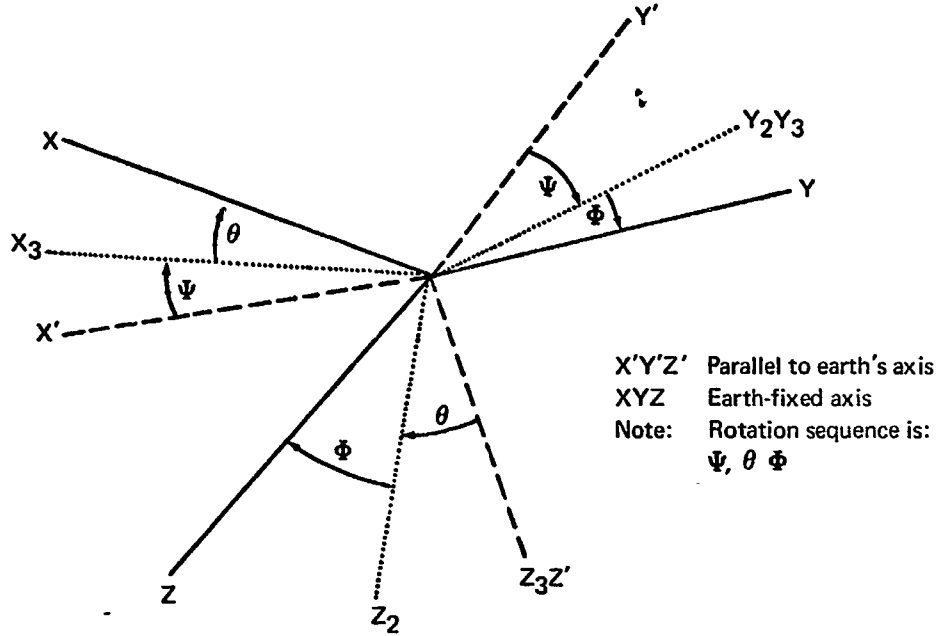


FIGURE 4. – EULER ANGLES

The gravity force vector is in the direction of  $z'$ , i.e.,  $\vec{\sigma} = g\vec{k}'$ . The transformation of  $g\vec{k}'$  to body-fixed axis is given by

$$g\vec{k}' = -g \sin \theta \vec{i} + g \cos \theta \sin \phi \vec{j} + g \cos \theta \cos \phi \vec{k}$$

Now, the general equations of motion may be written as six scalar equations in terms of the body axis system as

$$\dot{M}U + M(QW - RV) = -Mg \sin \theta + F_{A_x} + F_{T_x} \quad (5a)$$

$$M\dot{V} + M(RU + PW) = Mg \cos \theta \sin \phi + F_{A_y} + F_{T_y} \quad (5b)$$

$$M\dot{W} + M(PV + QU) = Mg \cos \theta \cos \phi + F_{A_z} + F_{T_z} \quad (5c)$$

$$I_{xx} \dot{P} - I_{xz} (\dot{R} + QP) + (I_{zz} - I_{yy}) QR = M_{A_x} + M_{T_x} \quad (5d)$$

$$I_{yy} \dot{Q} + I_{xz} (P^2 - R^2) + (I_{zz} - I_{yy}) QR = M_{A_{y_1}} + M_{T_{y_1}} \quad (5e)$$

$$I_{zz} \dot{R} - I_{xz} (\dot{P} - QR) + (I_{xx} - I_{zz}) PR = M_{A_z} + M_{T_z} \quad (5f)$$

These are six equations in the eight unknown quantities,  $U, V, W, P, Q, R, \phi$ , and  $\theta$ . The aerodynamic and thrust forces and moments in the right-hand members are functions of these eight quantities or their derivatives with respect to time and, possibly, control variables. The control variables are regarded as known functions of time.

A complete set of equations is obtained by introducing the following two kinematic relations obtained from the Euler angle definitions:

$$\dot{\theta} = Q \cos \phi - R \sin \phi \quad (6a)$$

$$\dot{\phi} = P + Q \sin \phi \tan \theta + R \cos \phi \tan \theta \quad (6b)$$

The heading angle for the airplane may be obtained from the additional kinematic relation

$$\dot{\psi} = (Q \sin \phi + R \cos \phi) \sec \theta \quad (7)$$

Equations (5) and (6) are eight equations in eight unknowns. They may be integrated if the aerodynamic and thrust terms are specified functions of the motion variables, but they are a nonlinear system of ordinary differential equations. If initial data, consisting of  $U, V, W, P, Q, R, \theta$ , and  $\phi$  specified at an initial instant of time  $t_0$  are given, then in general it is possible to integrate equations (5) and (6). This results in a determination of the variables at times later than  $t_0$ . Because of the nonlinearity of these equations, however, integration is possible only by mechanical quadratures, except for exceptional cases consisting of steady motion of the airplane.

Fortunately, the cases of steady motion that satisfy equations (5) and (6) represent solutions that are of prime interest. They are the subject of the following paragraph. The stability of an airplane in these steady motions, i.e., the tendency for the airplane to persist in the steady motion when disturbed from it, is evaluated using perturbation forms of the general equations of motion, equations (5) and (6). These perturbation forms are discussed in pars. 5.1.3 and 5.1.4. Paragraph 5.1.4 also introduces expressions that describe the dependence of the aerodynamics on the motion variables.

5.1.2 Steady reference motion.— In a steady reference motion all of the derivatives with respect to time in equations (5) vanish. The equations of motion must be satisfied for the airplane to be in equilibrium. Therefore, the steady velocities (denoted with a subscript one as  $U_1, V_1, W_1, P_1, Q_1, R_1$ ) must satisfy

$$M(Q_1 W_1 - R_1 V_1) = -Mg \sin \theta_1 + F_{A_{x_1}} + F_{T_{x_1}} \quad (8a)$$

$$M(R_1 U_1 - P_1 W_1) = Mg \cos \theta_1 \sin \phi_1 + F_{A_{y_1}} + F_{T_{y_1}} \quad (8b)$$

$$M(P_1 V_1 - Q_1 U_1) = Mg \cos \theta_1 \cos \phi_1 + F_{A_{z_1}} + F_{T_{z_1}} \quad (8c)$$

$$-I_{xz} Q_1 P_1 + (I_{zz} - I_{yy}) Q_1 R_1 = M_{A_{x_1}} + M_{T_{x_1}} \quad (8d)$$

$$I_{xz} (P_1^2 - R_1^2) + (I_{xx} - I_{zz}) P_1 R_1 = M_{A_{y_1}} + M_{T_{y_1}} \quad (8e)$$

$$I_{xz} Q_1 R_1 + (I_{yy} - I_{xx}) P_1 Q_1 = M_{A_{z_1}} + M_{T_{z_1}} \quad (8f)$$

and the kinematic relations

$$\dot{\theta}_1 = Q_1 \cos \phi_1 - R_1 \sin \phi_1 \quad (9a)$$

$$\dot{\phi}_1 = P_1 + (Q_1 \sin \phi_1 + R_1 \cos \phi_1) \tan \theta_1 \quad (9b)$$

Rates of change of the Euler angles  $\dot{\theta}_1$  and  $\dot{\phi}_1$  have been admitted, but it must be recognized that this leads to time rates of change in the components of gravity force in the right-hand members of equations (8a), (8b), and (8c). The left-hand members of these equations are invariant in time. Thus, in the case of motion in which  $\dot{\theta}_1$  and  $\dot{\phi}_1$  are nonzero, to maintain steady motion the aerodynamic forces and moments, or those due to thrust, must be controlled in such a manner as to balance the gravity force changes. In all cases for which  $\dot{\theta}_1$  and  $\dot{\phi}_1$  are zero the aerodynamic and thrust terms are constant during the steady motion.



Four of the most important steady reference flight conditions for assessing the stability and control characteristics of large airplanes are level flight, climbing flight, turning flight, and pullup. The steady velocity components and Euler angles and the equations of motion for those four cases are as follows:

a. Steady, level, rectilinear flight:

$$\begin{aligned}
 V_1 = P_1 = Q_1 = R_1 = \dot{\phi}_1 = 0 \quad , \quad \theta_1 = \text{constant} \\
 -Mg \sin \theta_1 + F_{A_{x_1}} + F_{T_{x_1}} &= 0 \\
 F_{A_{y_1}} + F_{T_{y_1}} &= 0 \\
 Mg \cos \theta_1 + F_{A_{z_1}} + F_{T_{z_1}} &= 0 \\
 M_{A_{x_1}} + M_{T_{x_1}} &= 0 \\
 M_{A_{z_1}} + M_{T_{z_1}} &= 0
 \end{aligned} \tag{10}$$

b. Steady, climbing flight:

The equations are identical to equations (10), although it must be noted that atmospheric density variation will lead to unsteady motion for constant control settings. However, this unsteady motion may be regarded as negligible for shallow climbs over moderate time periods.

c. Steady, turning flight:

$$\begin{aligned}
 V_1 = 0 \quad , \quad \theta_1 = \text{constant} \quad \text{and} \quad \phi_1 = \text{constant} \\
 MQ_1 W_1 = -Mg \sin \theta_1 + F_{A_{x_1}} + F_{T_{x_1}} \\
 M(R_1 U_1 - P_1 W_1) = Mg \cos \theta_1 \sin \phi_1 + F_{A_{y_1}} + F_{T_{y_1}} \\
 -MQ_1 U_1 = Mg \cos \theta_1 \cos \phi_1 + F_{A_{z_1}} + F_{T_{z_1}} \\
 -I_{xz} Q_1 P_1 + (I_{zz} - I_{yy}) Q_1 R_1 = M_{A_{x_1}} + M_{T_{x_1}} \\
 I_{xz} (P_1^2 - R_1^2) + (I_{xx} - I_{zz}) P_1 R_1 = M_{A_{y_1}} + M_{T_{y_1}} \\
 I_{xz} Q_1 R_1 + (I_{yy} - I_{xx}) P_1 Q_1 = M_{A_{z_1}} + M_{T_{z_1}}
 \end{aligned} \tag{11}$$

where the kinematic relations, equations (9), may be used to eliminate  $Q_1$  and  $P_1$  by introducing

$$\begin{aligned} Q_1 &= R_1 \tan \phi_1 \\ P_1 &= -R_1 \frac{\tan \theta_1}{\cos \phi_1} \end{aligned} ;$$

d. Steady pull-up:

$$\begin{aligned} V_1 = P_1 = R_1 &= 0 \quad \text{and} \quad \phi_1 = 0 \\ MQ_1 W_1 &= -Mg \sin \theta_1 + F_{A_{x_1}} + F_{T_{x_1}} \\ F_{A_{y_1}} + F_{T_{y_1}} &= 0 \\ -MQ_1 U_1 &= Mg \cos \theta_1 + F_{A_{z_1}} + F_{T_{z_1}} = 0 \\ M_{A_{x_1}} + M_{T_{x_1}} &= 0 \\ M_{A_{y_1}} + M_{T_{y_1}} &= 0 \\ M_{A_{z_1}} + M_{T_{z_1}} &= 0 \end{aligned} \tag{12}$$

The kinematic relation, equation (9a), gives

$$\dot{\theta}_1 = Q_1$$

To maintain a steady pull-up it is necessary that

$$Mg \cos \theta_1 Q_1 = \dot{F}_{A_{x_1}} + \dot{F}_{T_{x_1}}$$

and

$$Mg \sin \theta_1 Q_1 = \dot{F}_{A_{z_1}} + \dot{F}_{T_{z_1}}$$

5.1.3 Perturbation equations of motion.— Many of the major objectives of a stability and control evaluation of an airplane are achieved by considering motion deviating from a specified steady reference motion of the airplane. This deviation is often termed a perturbation, and the variables that describe the motion are taken to have values consisting of the sum of the reference value and a perturbation value. The notation of par. 5.1.2 is used to identify the reference values of the variables, while a lowercase letter or a subscript p is used to identify the perturbation values, e.g.,

$$U = U_1 + u$$

$$\theta = \theta_1 + \theta_p$$

The perturbation equations of motion are obtained from the general equations of motion, equations (5) and (6), by replacing the velocities, accelerations, and Euler angles appearing in those equations by the sum of a perturbation and a reference value. The reference values correspond to steady motion so that their derivatives with respect to time vanish.

The substitution of perturbation and reference values for the values of the variables appearing explicitly in equations (5) and (6) does not constitute an approximation. However, an assumption is made regarding the aerodynamic and thrust terms. These terms are functions and are, in general, functions of the velocity of the airplane and its acceleration. It is assumed that the functionality is such that it is possible to write

$$\begin{aligned} \vec{F}_A &= \vec{F}_{A_1} + \vec{f}_A, & \vec{F}_T &= \vec{F}_{T_1} + \vec{f}_T \\ \vec{M}_A &= \vec{M}_{A_1} + \vec{m}_A, & \vec{M}_T &= \vec{M}_{T_1} + \vec{m}_T \end{aligned} \quad (13)$$

Clearly, this eliminates functionality that is transcendental as in the case of the gravity forces that have transcendental (harmonic) dependence on the Euler angles.

The perturbation form for the equations of motion may now be written as

$$\begin{aligned} M\dot{u} + M[Q_1 W_1 - R_1 V_1 + Q_1 w + W_1 q + q\dot{w} - R_1 v - V_1 r - rv] \\ = -Mg \sin(\theta_1 + \theta_p) + F_{A_{x_1}} + F_{T_{x_1}} + f_{A_x} + f_{T_x} \end{aligned} \quad (14a)$$

$$\begin{aligned}
M\dot{v} + M(R_1 U_1 - P_1 W_1 + R_1 u + U_1 r + ur - P_1 w - W_1 p - wp) \\
= Mg \cos(\theta_1 + \theta_p) \sin(\phi_1 + \phi_p) + F_{A_{y_1}} + F_{T_{y_1}} + f_{A_y} + f_{T_y}
\end{aligned} \tag{14b}$$

$$\begin{aligned}
M\dot{w} + M(P_1 V_1 - Q_1 U_1 + P_1 v + V_1 p + pv - Q_1 u - U_1 q - qu) \\
= Mg \cos(\theta_1 + \theta_p) \cos(\phi_1 + \phi_p) + F_{A_{z_1}} + F_{T_{z_1}} + f_{A_z} + f_{T_z}
\end{aligned} \tag{14c}$$

$$\begin{aligned}
\textcircled{G11} \quad I_{xx} \dot{p} - I_{xz} (\dot{r} + Q_1 p + P_1 q + pq) + (I_{zz} - I_{yy}) (Q_1 r + R_1 q + rq) \\
= m_{A_x} + m_{T_x}
\end{aligned} \tag{14d}$$

$$\begin{aligned}
I_{yy} \dot{q} + I_{xz} (2P_1 p + p^2 - 2R_1 r - r^2) + (I_{xx} - I_{zz}) (P_1 r + R_1 p + rp) \\
= m_{A_y} + m_{T_y}
\end{aligned} \tag{14e}$$

$$\begin{aligned}
I_{zz} \dot{r} - I_{xz} (\dot{p} - Q_1 r - R_1 q - qr) + (I_{yy} - I_{xx}) (P_1 q + Q_1 p + pq) \\
= m_{A_z} + m_{T_z}
\end{aligned} \tag{14f}$$

Equations (14) are called the arbitrary perturbation equations of motion. The term "arbitrary" is used because no restrictions have been placed on the admissible magnitudes of the perturbations. They govern the motion of a rigid airplane to the same order of approximation as the general equations of motion, equations (5), except that the aerodynamic and thrust terms have been separated into reference and perturbation terms. That separation led to a simplification of equations (14d), (14e), and (14f). Since the values of the reference motion variables satisfy the equations of motion themselves, those terms involving only reference motion vanish. This simplification is not possible in equations (14a), (14b), and (14c) because of the sine and cosine functional dependence of the gravity force components.

Large perturbation equations of motion are obtained by restricting perturbations of the Euler angles  $\theta$  and  $\phi$  to a magnitude less than about 7.5 deg. Then the trigonometric functions of these angles are approximated by

$$\begin{aligned}
\cos \theta_p &\approx 1 & \sin \theta_p &\approx \theta_p \\
\cos \phi_p &\approx 1 & \sin \phi_p &\approx \phi_p
\end{aligned}
\tag{15} \textcircled{G9}$$

Introducing the approximations of equations (15) into equations (14a), (14b), and (14c) leads to

$$\begin{aligned}
M\dot{u} + M(Q_1 w + W_1 q + q w - R_1 v - V_1 r - r v) \\
= Mg \theta_p \cos \theta_1 + f_{A_x} + f_{T_x}
\end{aligned}
\tag{16a}$$

$$\begin{aligned}
M\dot{v} + M(R_1 u + U_1 r + u r - P_1 w - W_1 p - w p) \\
= Mg (-\theta_p \sin \theta_1 \sin \phi_1 + \phi_p \cos \phi_1 \cos \theta_1) + f_{A_y} + f_{T_y}
\end{aligned}
\tag{16b}$$

$$\begin{aligned}
M\dot{w} + M(P_1 v + V_1 p + p v - Q_1 u - U_1 q - q u) \\
= -Mg (\theta_p \sin \theta_1 \cos \phi_1 + \phi_p \sin \phi_1 \cos \theta_1) + f_{A_z} + f_{T_z}
\end{aligned}
\tag{16c}$$

Equations (16) along with equations (14d), (14e), and (14f) represent large perturbation equations of motion. It should be noted that terms involving products of  $\theta_p$  and  $\phi_p$  have been deleted. This is consistent with the approximations represented by equations (15), which are truncated power series expansions of sine and cosine functions. The system of equations is linear in terms of the perturbation Euler angles but nonlinear in the perturbation velocities.

Small perturbation equations of motion are obtained by deleting products of the perturbation motion variables. Equations (14) reduce to

$$M\dot{u} + M(Q_1 w + W_1 q - R_1 v - V_1 r) = Mg \theta_p \cos \theta_1 + f_{A_x} + f_{T_x}
\tag{17a}$$

$$M\dot{V} + M(P_1 r - P_1 w - W_1 p) = Mg(-\theta_p \sin \theta_1 \sin \phi_1 + \phi_p \cos \theta_1) + f_{A_y} + f_{T_y} \quad (17b)$$

$$M\dot{W} + M(P_1 v + V_1 p - Q_1 u - U_1 q) = -Mg(\theta_p \sin \theta_1 \cos \phi_1 + \phi_p \sin \phi_1 \cos \theta_1) + f_{A_z} + f_{T_z} \quad (17c)$$

(G5)

$$I_{xx} \dot{p} - I_{xz} (\dot{r} + Q_1 p + P_1 q) + (I_{zz} - I_{yy}) (Q_1 r + R_1 q) = m_{A_x} + m_{T_x} \quad (17d)$$

$$I_{yy} \dot{q} + 2 I_{xz} (P_1 p - R_1 r) + (I_{xx} - I_{zz}) (P_1 r + R_1 p) = m_{A_y} + m_{T_y} \quad (17e)$$

$$I_{zz} \dot{r} - I_{xz} (\dot{p} - Q_1 r - R_1 q) + (I_{yy} - I_{xx}) (P_1 q + Q_1 p) = m_{A_z} + m_{T_z} \quad (17f)$$

and the kinematic relations, equations (9), become

$$\dot{\theta}_p = -Q_1 \phi_p \sin \phi_1 + q \cos \phi_1 - R_1 \phi_p \cos \phi_1 - r \sin \phi_1 \quad (18a)$$

(G8)

$$\dot{\phi}_p = p + (Q_1 \phi_p \cos \phi_1 + r \sin \phi_1 - R_1 \phi_p \sin \phi_1$$

$$+ q \cos \phi_1) \tan \theta_1 + (Q_1 \sin \phi_1 + R_1 \cos \phi_1) \phi_p \quad (18b)$$

The terms in the small perturbation equations of motion appear to be of consistent order of magnitude. They are a set of linear ordinary differential equations provided the aerodynamic and thrust terms are written as linear functions of the perturbation motion variables. The form of the aerodynamic and thrust terms is the subject of par. 5.1.4 and app. B.

An additional simplification is possible if consideration is given to the orders of magnitude of the reference motion variables. In the analysis of airplane response, the perturbation motion variables have small but finite values. The limits on the validity of the application of the small perturbation equations are set primarily by the limit of validity of linear aerodynamic theory. In many cases, this limitation may apply equally to the prediction of the aerodynamic forces associated with the reference motion; hence, in these cases the reference motion rotation rates and sideslip velocity must be limited to the same orders of magnitude as those of the perturbation variables.

Equations (17) apply to the case of stability and control analysis when the aerodynamic terms are obtained from a linearization of the aerodynamics about some condition where the aerodynamic phenomena are basically nonlinear. The perturbation motion variables in this case are then limited in magnitude by the range of validity of the linearization. Aerodynamic terms in this case are obtained empirically from test data.

Many important stability and control evaluations are performed on cases where the magnitudes of the reference motion variables are sufficiently small to permit their products with the perturbation variables to be neglected. This is the case where stability and control characteristics of transport aircraft in their cruise condition are concerned. For these cases the small perturbation equations may be reduced to

$$M\dot{u} + MW_1 q = Mg \theta_p \cos \theta_1 + f_{A_x} + f_{T_x} \quad (19a)$$

$$M\dot{v} - MW_1 p = -Mg(\theta_p \sin \theta_1 \sin \phi_1 - \phi_p \cos \phi_1 \cos \theta_1) + f_{A_y} + f_{T_y} \quad (19b)$$

$$M\dot{w} + MU_1 q = -Mg(\theta_p \sin \theta_1 \cos \phi_1 + \phi_p \sin \phi_1 \cos \theta_1) + f_{A_z} + f_{T_z} \quad (19c) \textcircled{D4}$$

$$I_{xx}\dot{p} - I_{xz}\dot{r} = m_{A_x} + m_{T_x} \quad (19d)$$

$$I_{yy}\dot{q} = m_{A_y} + m_{T_y} \quad (19e)$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} = m_{A_z} + m_{T_z} \quad (19f)$$

and the kinematic relations reduce to

$$\theta_p = q \cos \theta_1 - r \sin \phi_1 \quad (20a)$$

$$\phi_p = p + (r \sin \phi_1 + q \cos \phi_1) \tan \theta_1 \quad (20b)$$

D4

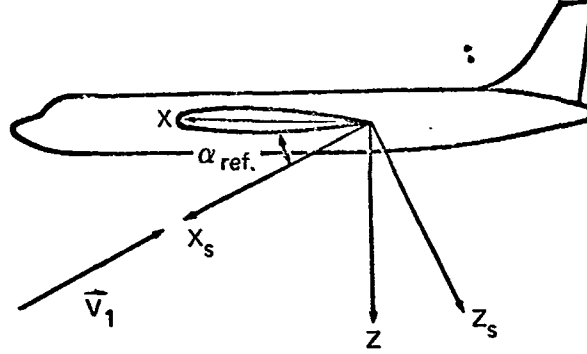
**5.1.4 Aerodynamic derivatives.**— The concept of an aerodynamic derivative or, more commonly, a stability derivative is developed in app. B. It relates a component of aerodynamic force or moment acting on an airplane to a motion variable or a parameter that describes a change in airplane shape such as a control surface deflection angle. The change in airplane lift with change in airplane angle of attack is expressed as

$$C_{L_\alpha} \equiv \frac{1}{qS} \frac{\partial L}{\partial \alpha}$$

The notation of a partial derivative is appropriate, since all other motion variables and shape parameters are held constant while the angle of attack is allowed to vary. Besides, the lift is not, in general, a linear function of angle of attack or any of the other variables that influence the airplane lift. The stability derivative  $C_{L_\alpha}$ , therefore, varies with the airplane's flight condition. For perturbation motion of the airplane about a reference motion, the stability derivative is evaluated for the reference motion and may be considered to be constant in the perturbation motion. This local linearization, discussed more fully in Sec. 7 of this volume and in detail in app. B, is a widely accepted practice (refs. 4 and 6). The basic mathematical theory is given by reference 7.

Another widely accepted practice is to introduce a stability axis system (ref. 4, p. 103). This is a body-fixed-axis system having  $X_s$  forward and in the direction of the freestream velocity in the reference motion of the airplane. The body axis system  $X,Y,Z$  is rotated about the  $Y$  axis from the stability axis system by an angle  $\alpha_{ref}$ , as shown in the following sketch:





where  $\vec{V}_1$  is the freestream velocity vector in the reference motion so that  $\vec{V}_1 = -\vec{V}_{c1}$ . The purpose of the stability axis system is to introduce a simplification in the equations of motion, which is discussed in ref. 4 but is not used in this study.

The reference angle of attack is used but is denoted as  $\alpha_1$ , i.e., the angle of attack in the reference flight condition. Lift and drag of the airplane are measured in a wind axis system such that in reference flight

$$F_{A_{x_1}} = -D_1 \cos \alpha_1 + L_1 \sin \alpha_1$$

and

$$F_{A_{z_1}} = -D_1 \sin \alpha_1 - L_1 \cos \alpha_1$$

In the perturbed motion

$$F_{A_z} = -D \cos (\alpha_1 + \alpha) + L \sin (\alpha_1 + \alpha)$$

$$F_{A_z} = -D \sin (\alpha_1 + \alpha) - L \cos (\alpha_1 + \alpha)$$

Using these relations and the small angle approximation such that

$$\sin (\alpha_1 + \alpha) \approx \sin \alpha_1 + \alpha \cos \alpha_1$$

and

$$\cos (\alpha_1 + \alpha) \approx \cos \alpha_1 - \alpha \sin \alpha_1$$

the perturbation aerodynamic forces

$$\begin{aligned} f_{A_x} = & -\cos \alpha_1 D_p + \sin \alpha_1 D_1 \alpha + \cos \alpha_1 L_1 \alpha + \sin \alpha_1 L_p \\ & + \sin \alpha_1 D_p \alpha + \cos \alpha_1 L_p \alpha \end{aligned} \quad (21)$$

$$\begin{aligned} f_{A_z} = & -\cos \alpha_1 D_1 \alpha - \sin \alpha_1 D_p - \cos \alpha_1 L_p + \sin \alpha_1 L_1 \alpha \\ & - \cos \alpha_1 D_p \alpha + \sin \alpha_1 L_p \alpha \end{aligned}$$

for small perturbations, products of perturbation quantities are neglected so that

$$\begin{aligned} f_{A_x} = & -\cos \alpha_1 D_p + \sin \alpha_1 D_p \alpha + \cos \alpha_1 L_1 \alpha + \sin \alpha_1 L_p \\ f_{A_z} = & -\cos \alpha_1 D_1 \alpha - \sin \alpha_1 D_p - \cos \alpha_1 L_p + \sin \alpha_1 L_1 \alpha \end{aligned} \quad (22)$$

The perturbation components of aerodynamic force and moment are written in terms of stability derivatives as (app. B)

$$\begin{aligned} D_p = \bar{q}_1 S_w \left[ \left( C_{D_u} + 2 C_{D_1} \right) \frac{u}{V_{c_1}} + C_{D_\beta} |\beta| + C_{D_{\dot{\beta}}} \left| \frac{\dot{\beta} b}{2 V_{c_1}} \right| \right. \\ \left. + C_{D_\alpha} \alpha + C_{D_{\dot{\alpha}}} \frac{\dot{\alpha} \bar{c}}{2 V_{c_1}} + C_{D_p} \left| \frac{pb}{2 V_{c_1}} \right| + C_{D_q} \frac{q \bar{c}}{2 V_{c_1}} \right. \\ \left. + C_{D_r} \left| \frac{rb}{2 V_{c_1}} \right| + \sum_i C_{D_{\delta_i}} \delta_i \right] \end{aligned} \quad (23a)$$

$$\begin{aligned}
L_p = \bar{q}_1 S_w \left[ \left( C_{L_u} + 2C_{L_1} \right) \frac{u}{V_{c_1}} + C_{L_\beta} |\beta| + C_{L_{\dot{\beta}}} \left| \frac{\dot{\beta} b}{2V_{c_1}} \right| \right. \\
+ C_{L_\alpha} \alpha + C_{L_{\dot{\alpha}}} \frac{\dot{\alpha} \bar{c}}{2V_{c_1}} + C_{L_p} \left| \frac{pb}{2V_{c_1}} \right| + C_{L_q} \frac{qc}{2V_{c_1}} \\
\left. + C_{L_r} \left| \frac{rb}{2V_{c_1}} \right| + \sum_i C_{L_{\delta_i}} \delta_i \right]
\end{aligned} \tag{23b}$$

$$\begin{aligned}
f_{A_y} = \bar{q}_1 S_w \left[ 2C_{y_1} \frac{u}{V_{c_1}} + C_{y_\beta} \beta + C_{y_{\dot{\beta}}} \frac{\dot{\beta} b}{2V_{c_1}} + C_{y_p} \frac{pb}{2V_{c_1}} \right. \\
\left. + C_{y_r} \frac{rb}{2V_{c_1}} + \sum_i C_{y_{\delta_i}} \delta_i \right]
\end{aligned} \tag{23c}$$

$$\begin{aligned}
m_{A_x} = \bar{q}_1 S_w b \left[ 2C_{\ell_1} \frac{u}{V_{c_1}} + C_{\ell_\beta} \beta + C_{\ell_{\dot{\beta}}} \frac{\dot{\beta} b}{2V_{c_1}} + C_{\ell_p} \frac{pb}{2V_{c_1}} \right. \\
\left. + C_{\ell_r} \frac{rb}{2V_{c_1}} + \sum_i C_{\ell_{\delta_i}} \delta_i \right]
\end{aligned} \tag{23d}$$

$$\begin{aligned}
m_{A_y} = \bar{q}_1 S_w \bar{c} \left[ \left( C_{m_u} + 2C_{m_1} \right) \frac{u}{V_{c_1}} + C_{m_\beta} |\beta| + C_{m_{\dot{\beta}}} \left| \frac{\dot{\beta} b}{2V_{c_1}} \right| \right. \\
+ C_{m_\alpha} \alpha + C_{m_{\dot{\alpha}}} \frac{\dot{\alpha} \bar{c}}{2V_{c_1}} + C_{m_p} \left| \frac{pb}{2V_{c_1}} \right| + C_{m_q} \frac{qc}{2V_{c_1}} \\
\left. + C_{m_r} \left| \frac{rb}{2V_{c_1}} \right| + \sum_i C_{m_{\delta_i}} \delta_i \right]
\end{aligned} \tag{23e}$$

$$\begin{aligned}
m_{A_z} = \bar{q}_1 S_w b \left[ 2C_{n_1} \frac{u}{V_{c_1}} + C_{n_\beta} \beta + C_{n_{\dot{\beta}}} \frac{\dot{\beta} b}{2V_{c_1}} \right. \\
\left. + C_{n_p} \frac{pb}{2V_{c_1}} + C_{n_r} \frac{rb}{2V_{c_1}} + \sum_i C_{n_{\delta_i}} \delta_i \right]
\end{aligned} \tag{23f}$$

The perturbation components of thrust forces and moments are

$$\begin{aligned}
 f_{T_x} &= \bar{q}_1 S_w \left( 2 C_{T_{x_1}} \frac{u}{V_{c_1}} + \sum_j C_{T_{x_{\eta_j}}} \eta_j \right) \\
 f_{T_y} &= \bar{q}_1 S_w \left( 2 C_{T_{y_1}} \frac{u}{V_{c_1}} + \sum_j C_{T_{y_{\eta_j}}} \eta_j \right) \\
 f_{T_z} &= \bar{q}_1 S_w \left( 2 C_{T_{z_1}} \frac{u}{V_{c_1}} + \sum_j C_{T_{z_{\eta_j}}} \eta_j \right) \\
 m_{T_x} &= \bar{q}_1 S_w b \left( 2 C_{T_{\ell_1}} \frac{u}{V_{c_1}} + \sum_j C_{T_{\ell_{\eta_j}}} \eta_j \right) \\
 m_{T_y} &= \bar{q}_1 S_w \bar{c} \left( 2 C_{T_{m_1}} \frac{u}{V_{c_1}} + \sum_j C_{T_{m_{\eta_j}}} \eta_j \right) \\
 m_{T_z} &= \bar{q}_1 S_w b \left( 2 C_{T_{n_1}} \frac{u}{V_{c_1}} + \sum_j C_{T_{n_{\eta_j}}} \eta_j \right)
 \end{aligned} \tag{23g}$$

The above results may be combined with the perturbation equations of motion of par. 5.1.3. The perturbation force components in the X and Z directions must be computed differently for large and small perturbations. Equations (21) must be used to relate perturbation lift and drag to the X and Z directions for large perturbations; for small perturbations equations (22) must be used.

## 5.2 Equations of Motion for a Flexible Airplane

**5.2.1 Lumped parameterization.**— In the preceding section, the motion of a rigid airplane was described in terms of six degrees of freedom of the airplane's center of gravity—three translational and three rotational. The effects of inertia and gravity can be completely accounted for in terms of forces and moments applied at the airplane's center of gravity. And, although the distribution of aerodynamic surface stresses depends on the shape of a rigid airplane, its dependence upon the rigid airplane's motion may be expressed entirely in terms of the motion of its center of gravity. Finally, the effects of the aerodynamic surface stresses on the rigid airplane's motion are accounted for by aerodynamic forces and moments acting at the airplane's center of gravity.

When an airplane is flexible, however, motion which is relative to the center of gravity must be considered. Equations of motion written entirely in terms of center of gravity motion variables and in terms of forces and moments considered to act at the center of gravity are no longer sufficient. The additional motion is called elastic motion, and it is common terminology to refer to "elastic" degrees of freedom.

The flexible airplane is essentially a continuous, elastic body. As such it has a continuous infinity of elastic degrees of freedom. Thus, unless some simplifying approximations are introduced the equations of motion for the elastic motion will be integrodifferential equations (app. A, equation (6.35)). To avoid this complication the airplane is divided into a large number of elements. Each element is regarded to have at most six degrees of freedom. If their number is  $n$ , the equations of motion of a flexible airplane will govern motion in  $6n$  degrees of freedom and be  $6n$  in number. Certain approximations will be introduced to reduce the number of equations, but the number of degrees of freedom will remain effectively unchanged. The motion in some degrees of freedom will be dependent motion. This is the subject of several later paragraphs in this volume as well as of much of app. A.

This paragraph, as well as several of those following, makes use of matrices in the formulation. When matrices of a particular type or matrix operations are introduced, or properties of a matrix are discussed, an article will be cited from Frazer, Duncan, and Collar (ref. 3). The reader may refer to the cited article for the appropriate explanation.

Letting the total volume of the airplane be represented by  $V$  and the volume of the  $i^{\text{th}}$  element by  $V_i$ , then

$$V = \sum_{i=1}^n V_i \quad (24)$$

The total mass of the airplane is  $M$  and the density is  $\rho_A$ . The mass of the  $i^{\text{th}}$  element is, therefore,

$$m_i = \int_{V_i} \rho_A dV \quad (25)$$

Also, recalling that  $\bar{r}$  is the position relative to the airplane's center of mass, the position of the  $i^{\text{th}}$  lumped mass is taken to be at the point

$$\bar{r}_i = \frac{1}{m_i} \int_{V_i} \bar{r} \rho_A dV \quad (26)$$

This is the centroid of  $m_i$  and, although refinements may be introduced, the density is taken to be uniform over the element of volume  $V_i$  so that

$$\bar{\mathbf{r}}_i = \frac{1}{V_i} \int_{V_i} \bar{\mathbf{r}} dV \quad (27)$$

which is the location of the geometric centroid.

The geometry of the flexible airplane is defined as shown by fig. 5.

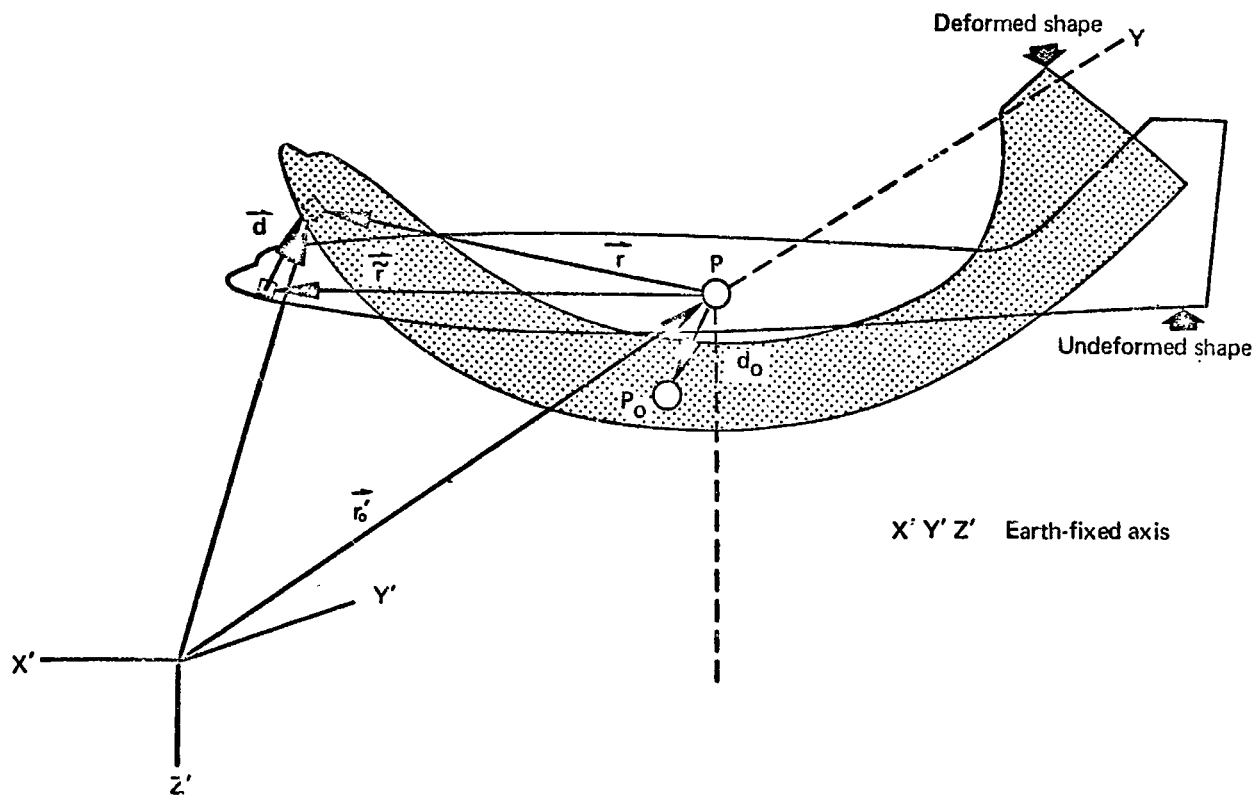


FIG. 5.— AXIS SYSTEM FOR THE DEFORMED SHAPE AND THE UNDEFORMED SHAPE OF AN ELASTIC AIRPLANE

The elastic displacement of a mass particle is a vector,  $\bar{\mathbf{d}}$ , which is a function of position and time. It describes the change in position relative to the airplane's center of

gravity. This elastic deformation carries each mass particle from its position,  $\tilde{\mathbf{r}}_i$ , in the undeformed airplane to its position,  $\mathbf{\tilde{r}}_i$ , in the deformed airplane. Thus, the elastic displacement vector is a function of position and time as

$$\tilde{\mathbf{d}} = \tilde{\mathbf{d}}(\tilde{\mathbf{r}}, t)$$

where it may be noted that the dependence on position has been written as a function of its undeformed position,  $\tilde{\mathbf{r}}$ . This is referred to as a Lagrangian description (ref. 8, p. 29).

For the lumped parameter formulation each volume element is taken to displace as a whole with the mean displacement given by

$$\tilde{\mathbf{d}}_i = \frac{1}{m_i} \int_{V_i} \rho_A \tilde{\mathbf{d}} dV \quad (28)$$

The totality of all elastic displacements is written as a column matrix (ref. 3, Art. 1.2) as

$$\{\mathbf{d}\}^T = [d_{x_1}, d_{y_1}, d_{z_1}, d_{x_2}, d_{y_2}, d_{z_2}, \dots, d_{x_n}, d_{y_n}, d_{z_n}] \quad (29)$$

Elastic rotations relative to the airplane as a whole occur for a mass particle and are given as

$$\frac{1}{2} \nabla \times \tilde{\mathbf{d}}$$

When the airplane structure is regarded as a continuous body, the rotation of the mass particles does not contribute to the inertia forces. When the airplane is represented by a collection of elements with lumped masses, it is tempting to include rotational degrees of freedom and the rotational inertias of the lumped masses. But on consideration of the practical numerical problem of predicting the motion of the lumped masses, it is better to neglect rotation and include only the inertia forces generated by their mean translational motions.

The number of dynamically participating degrees of freedom that may be included in a numerical analysis is limited. Let the upper limit of that number be  $m$ . If rotational degrees of freedom are included dynamically, the total number of lumped masses is  $m/6$ . If they are ignored, the number of lumped masses may be twice as great, i.e.,  $m/3$ . The larger number of lumped masses more closely represents the continuous airplane. The smaller number must have mean rotations that are not readily defined as well as moments and products of inertia

that are even more difficult to define rationally. Therefore, the formulation that ignores the dynamics of mean rotations of lumped masses is the better approximation.

It must be clearly understood that rotational motion of the lumped masses has not been set to zero in the above. In fact, the aerodynamic surface stresses are dependent upon the surface slope of the airplane, and the surface slope changes as the airplane deforms elastically. The lumped surface slope (or mean slope) will be related to a mean rotation of the element, so that these quantities enter the problem through the aerodynamics even though their dynamic influence is neglected.

It is desirable to have a convenient method of computing the mass properties of the entire airplane from the values of the lumped masses and their positions relative to the center of gravity of the airplane,  $\vec{r}_i$ . To do this consider the manner in which the moments and products of inertia are introduced in mechanics, i.e., as a consequence of the integral expression for angular momentum about the center of gravity:

$$\vec{\psi} \cdot \vec{\omega} = \int_V \vec{r} \times (\vec{\omega} \times \vec{r}) \rho_A dV$$

This expression may be replaced by

$$\begin{aligned} \vec{\psi} \cdot \vec{\omega} &= \sum_{i=1}^n \int_{V_i} \vec{r} \times (\vec{\omega} \times \vec{r}) \rho_A dV \\ &= \sum_{i=1}^n \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) m_i \end{aligned}$$

so that the entire angular momentum is replaced by the sum of the contributions to the angular momentum by each lumped mass. Similarly, the total linear momentum is

$$M \vec{V}_c = \sum_{i=1}^n m_i \vec{V}_c \quad (30)$$

Introduce the diagonal mass matrix (ref. 3, Art. 1.2),  $[\mathbf{m}]$ , as defined in app. A, par. 6.3.1, i.e.,



$$[\tilde{m}] = \begin{bmatrix} m_1 & & & & & \\ & m_1 & & & & \\ & & m_1 & & & \\ & & & m_2 & & \\ & & & & m_2 & \\ & & & & & m_2 \\ & & & & & \ddots \\ & & & & & & m_n \\ & & & & & & & m_n \\ & & & & & & & & m_n \end{bmatrix} \quad (31)$$

and the rigid-body mode shape matrix  $[\bar{\phi}]$  defined as

$$[\bar{\phi}] = \begin{bmatrix} [\bar{\phi}_1] \\ [\bar{\phi}_2] \\ \vdots \\ [\bar{\phi}_n] \end{bmatrix} \quad (32a) \quad \textcircled{\text{G10}}$$

where

$$[\bar{\phi}_i] = \begin{bmatrix} 1 & 0 & 0 & 0 & z_i & -y_i \\ 0 & 1 & 0 & -z_i & 0 & x_i \\ 0 & 0 & 1 & y_i & -x_i & 0 \end{bmatrix} \quad (32b)$$

Also, introduce the velocity matrix, written in transpose form (ref. 3, Art. 1.2) defined as

$$\{V\}^T = [U, V, W, P, Q, R] \quad (33)$$

so that components of momentum may be written as

$$\begin{bmatrix} M \vec{V}_c \cdot \vec{i} \\ M \vec{V}_c \cdot \vec{j} \\ M \vec{V}_c \cdot \vec{k} \\ (\vec{\psi} \cdot \vec{\omega}) \cdot \vec{i} \\ (\vec{\psi} \cdot \vec{\omega}) \cdot \vec{j} \\ (\vec{\psi} \cdot \vec{\omega}) \cdot \vec{k} \end{bmatrix} = [\bar{\phi}]^T [\bar{m}] [\bar{\phi}] \{V\} = [M] \{V\} \quad (34)$$

where the definition of the inertia matrix  $[M]$  is given by

$$[M] \equiv [\bar{\phi}]^T [\bar{m}] [\bar{\phi}] \quad (35)$$

Thus, the components of linear and angular momentum for the airplane are obtained in simple matrix expressions, equations (34) and (35).

The aerodynamic and thrust forces act at the external surface of the airplane. Let the total external surface,  $S$ , be subdivided into the external (exposed) surfaces of the lumped masses,  $S_i$ , so that

$$S = \sum_{i=1}^n S_i \quad (36)$$

Then the total aerodynamic and thrust forces on the airplane may, in accordance with equation (3) and this subdivision of the surface, be represented as

$$\begin{aligned} \vec{F}_A + \vec{F}_T &= \sum_{i=1}^n \int_{S_i} \vec{F} dS \\ &= \sum_{i=1}^n \vec{F}_i \end{aligned}$$

where  $\mathbf{F}_i$  is the force on the  $i^{\text{th}}$  lumped mass. Similarly, from equation (4)

$$\begin{aligned}\vec{\mathbf{M}}_A + \vec{\mathbf{M}}_T &= \sum_{i=1}^n \int_{S_i} \vec{\mathbf{r}} \times \vec{\mathbf{F}} \, dS \\ &= \sum_{i=1}^n \vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_i\end{aligned}$$

Now, introduce the matrix definitions of the components of the aerodynamic forces and moments:

$$\{\mathbf{F}_A\}^T = [F_{A_x}, F_{A_y}, F_{A_z}, M_{A_x}, M_{A_y}, M_{A_z}] \quad (37)$$

the components of the thrust forces and moments:

$$\{\mathbf{F}_T\}^T = [F_{T_x}, F_{T_y}, F_{T_z}, M_{T_x}, M_{T_y}, M_{T_z}] \quad (38)$$

and the components of element surface forces:

$$\{\mathbf{F}\}^T = [F_{x_1}, F_{y_1}, F_{z_1}, F_{x_2}, \dots, F_{x_n}, F_{y_n}, F_{z_n}] \quad (39)$$

Using the rigid-body mode shape matrix  $[\bar{\Phi}]$ , it follows that

$$\{\mathbf{F}_A\} + \{\mathbf{F}_T\} = [\bar{\Phi}]^T \{\mathbf{F}\} \quad (40)$$

so that a simple matrix expression relates the surface stresses to overall airplane forces and moments.

The results presented in this section form the basis for representing a continuous, flexible airplane as a collection of lumped masses. However, neither of two important details have as yet been considered. These are the dependence of elastic constraining forces, and the dependence of the aerodynamic forces, on the mean elastic displacements  $\{d\}$ . The elastic constraining forces are discussed in par. 5.2.2. The aerodynamics are discussed in Sec. 7.

The thrust forces may seem to have been slighted in the discussion. However, their direct effect in terms of forces applied at the engine attachment points may be readily introduced in an obvious manner and the induced effects on the aerodynamic forces must be accounted for in the stability derivatives.

**5.2.2 Structural flexibility.**— The flexibility of the airplane is fundamental to the considerations of this report. This paragraph introduces concepts to describe what is meant by flexibility in physical terms and presents the equations that describe it. An exposition of the theory underlying these equations is the subject of par. 4.2.3 of app. B. It suffices to note here that the underlying theory is that for infinitesimal deformation of a perfectly elastic, isotropic, and homogeneous solid. It is often called the classical theory of elasticity and is a linear approximation of an exact, nonlinear theory of elasticity.

The change in shape of an airplane when it is subjected to external loads is a manifestation of structural flexibility. The magnitude of the change, viz; deflection or deformation, is in approximately constant proportion to the load producing it. Thus, the deflection measured at a point A on the structure is linearly related to the load P at another point B. This is expressed as

$$\textcircled{\text{S1}} \quad \delta = CP \quad (41)$$

where the constant of proportionality, C, is termed the flexibility of the structure associated with the two points, A and B.

A general description of deformation follows by considering two positions of each point of the structure. They are the position  $\bar{r}$  before deformation and the position  $\bar{r}'$  after deformation, both of which are measured relative to the airplane's center of gravity. The deformation or displacement vector is given by

$$\bar{d} = \bar{r}' - \bar{r} \quad (42)$$

The displacement vector was originally introduced in par. 5.2.1, where the mean displacement of an element of the airplane was defined by equation (28) as

$$\bar{\mathbf{d}}_i = \frac{1}{m_i} \int_{V_i} \rho_A \bar{\mathbf{d}} dV$$

In the following, the mean displacement vector at the  $i^{\text{th}}$  element is related to the forces applied at the  $i^{\text{th}}$  element as well as all other elements.

The mean surface force on the  $j^{\text{th}}$  element is:

$$\bar{\mathbf{F}}_j = \frac{1}{S_j} \int_{S_j} \bar{\mathbf{F}} dS$$

An element of the airplane also experiences body forces such as inertia and gravity. These are proportional to the airplane density, and the mean values for the body forces are defined as

$$\bar{\mathbf{R}}_j = \int_{V_j} \rho_A \left( \frac{d^2 \bar{\mathbf{r}}}{dt^2} + \bar{\mathbf{g}} \right) dV$$

where  $d^2 \bar{\mathbf{r}} / dt^2$  is the acceleration relative to the earth-fixed reference system.

The mean displacement of the  $i^{\text{th}}$  element due to forces applied to the  $j^{\text{th}}$  element is defined as

$$\bar{\mathbf{d}}_i = -\bar{\mathbf{C}}_{o_{ij}} \cdot (\bar{\mathbf{R}}_j - \bar{\mathbf{F}}_j)$$

and the displacement of the  $i^{\text{th}}$  element due to forces at all elements is given by

$$\bar{\mathbf{d}}_i = \sum_{j=1}^n -\bar{\mathbf{C}}_{o_{ij}} \cdot (\bar{\mathbf{R}}_j - \bar{\mathbf{F}}_j)$$

Further, this expression is valid for all elements, so that there are  $n$  equations like (45b).

The subscript zero on the diadic  $\bar{C}_{oij}$  is used to indicate that the structure is clamped at the center of gravity, i.e., the point on the airplane that was at the c.g. before deformation is constrained against translation or rotation. That point may be allowed to translate as  $\bar{d}_o$  relative to a reference axis system with origin at the airplane's c.g. It may also be allowed to rotate through an angle  $\bar{\theta}_o$  relative to the reference axis system. Thus, the mean displacement  $\bar{d}_i$  relative to the reference axis system is given by

$$\bar{d}_i - \bar{d}_o - \bar{r}_i \times \bar{\theta}_o = - \sum_{j=1}^n \bar{C}_{oij} \cdot (\bar{R}_j - \bar{F}_j) \quad (46)$$

By defining the column matrix

$$\{B\}^T = [d_{x_o}, d_{y_o}, d_{z_o}, \theta_{x_o}, \theta_{y_o}, \theta_{z_o}] \quad (47)$$

the matrix definitions of par. 5.2.1 may be used to write equation (46) as

$$\{d\} - [\bar{\phi}] \{B\} = - [C_o] (\{R\} - \{F\}) \quad (48)$$

where the flexibility matrix (ref 3, Art. 8.11),  $[C_o]$ , is made up of  $n$  submatrices 3 by 3 in size and given by

$$\begin{bmatrix} C_{oxx_{ij}} & C_{oxy_{ij}} & C_{oxz_{ij}} \\ C_{oyx_{ij}} & C_{oyy_{ij}} & C_{oyz_{ij}} \\ C_{ozx_{ij}} & C_{ozy_{ij}} & C_{ozz_{ij}} \end{bmatrix}$$

so that the flexibility matrix  $[C_o]$  is  $3n$  by  $3n$  in size. The typical element  $C_{ozy_{ij}}$  is the component of deflection in the  $z$  direction at the  $i^{\text{th}}$  element due to a unit component of force in the  $y$  direction at the  $j^{\text{th}}$  element.

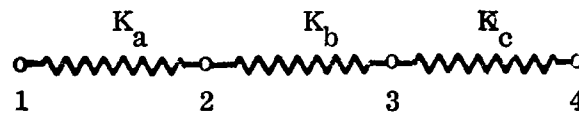
Equation (48) is the result required in the equations of motion of a flexible airplane represented by lumped parameters. However, the flexibility matrix must be computed from consideration of the structural details of the airplane. The flexibility matrix so derived will not be in the form defined for equation (48). The structural analysis treats the airplane structure as a continuous body. Thus, the flexibility matrix  $[C_0]$  must be obtained from a continuous analogue using the type of averaging process that led to definitions of mean displacements and forces in par. 5.2.1.

The problem of computing the flexibility matrix  $[C_0]$  is complicated by the fact that the structural analysis leads to a flexibility matrix that relates displacements at points to forces applied at points. The method of computation of  $[C_0]$  is not immediately obvious.

The structural analysis deals with an idealized structure made up of a collection of simple structural elements (app. B). For example, a typical wing structure is made up of panels of skin cut out by ribs and spars, the webs of the ribs and spars, as well as the rib and spar caps. The structural properties of each of these elements are known. The structural analysis proceeds by requiring the forces on the elements to be in equilibrium and by requiring the elements to remain joined together under load, i.e., continuity. The problem arises from the fact that this leads to a flexibility matrix in terms of forces and displacements at the points where the structural elements are joined.

Joining points appears inaccurate for the joining of some elements, e.g., a spar cap and a spar web. These elements obviously join along a line. The structural analysis will not satisfy continuity along the entire line of intersection. It will satisfy continuity only at a finite number of points, called node points, on the intersection.

One method of structural analysis, the displacement method, is readily illustrated by a simple one-dimensional example. For greater detail, reference is made to app. B. Consider an assemblage of springs:



The springs have stiffnesses represented by  $K_a$ ,  $K_b$ , and  $K_c$ , and the nodes are numbered as shown. There will be four nodal forces,  $F_i$  and four nodal displacements,  $d_i$ . Even though forces exist in the springs at all points between nodes and all interior points undergo displacement, only the nodal values will be required for the structural analysis.

The element stiffness matrix for the spring from node 1 to node 2 is a 2 by 2 matrix. The first column is the forces at the nodes for a unit displacement of node 1 to the right, i.e.,  $F_1 = K_a$ ,  $F_2 = -K_a$ . The second column is the forces at the nodes for a unit displacement of node 2 to the right, i.e.,  $F_1 = -K_a$ ,  $F_2 = K_a$ , so that the element stiffness is given by

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} K_a & -K_a \\ -K_a & K_a \end{bmatrix}$$

Similarly, two other element stiffness matrices may be written as

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} = \begin{bmatrix} K_b & -K_b \\ -K_b & K_b \end{bmatrix}$$

and

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} = \begin{bmatrix} K_c & -K_c \\ -K_c & K_c \end{bmatrix}$$

The stiffness matrix for the entire assemblage is obtained by forming a composite matrix as



$$[K] = \begin{bmatrix} \begin{bmatrix} K_a & -K_a \\ -K_a & (K_a + K_b) \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} (K_b + K_c) & -K_c \\ -K_c & K_c \end{bmatrix} \end{bmatrix}$$

and may be used to write

$$\{F\} = [K] \{d\} \quad (49)$$

or

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} K_a & -K_a & 0 & 0 \\ -K_a & K_a + K_b & -K_b & 0 \\ 0 & -K_b & K_b + K_c & -K_c \\ 0 & 0 & -K_c & K_c \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

The stiffness matrix  $[K]$  is singular (ref 3, Art. 1.8). Thus, the displacements are not uniquely related to the forces. This has to be the case, since an equal addition to each of the displacements does not cause a change in the forces. That addition would represent a translation as a rigid body.

The structure may be clamped at node 4 such that  $d_4 = 0$ . Now, the displacements are determinant. Let the matrices be partitioned (ref. 3, Art. 1.7), as

$$\begin{bmatrix} \{F_e\} \\ F_4 \end{bmatrix} = \begin{bmatrix} K_{11} & | & K_{14} \\ \hline K_{41} & | & K_{44} \end{bmatrix} \begin{bmatrix} \{d_e\} \\ 0 \end{bmatrix}$$

so that

$$\begin{aligned} \{F_e\} &= [K_{11}] \{d_e\} \\ F_4 &= [K_{41}] \{d_e\} \end{aligned} \quad (50)$$

The reduced stiffness matrix  $[K_{11}]$  has an inverse (ref 3, Art. 1.11), i.e.,

$$[K_{11}]^{-1}$$

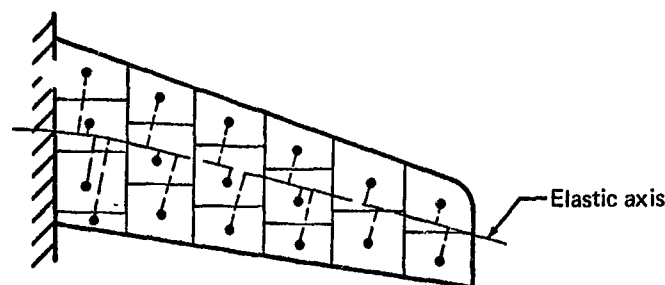
and it follows that

$$\{d_e\} = [K_{11}]^{-1} \{F_e\} \quad (51)$$

Finding the composite stiffness matrix to be singular in the above example is typical. The composite stiffness matrix for an entire airplane is also singular. The singularity is removed by setting enough nodal displacements to zero that the airplane structure is constrained against translation and rotation. The requirement is satisfied if a single point is clamped against rotation and translation. When this has been done the flexibility matrix may be obtained by inverting the reduced stiffness matrix. The displacements predicted by this matrix are changes in position relative to a coordinate system with origin fixed at the clamped point.

The example illustrates the use of element stiffnesses to describe the elastic properties of a structure in terms of forces and displacements at the nodes. The forces applied to the airplane are distributed. Also, the displacements of interest are mean displacements. Thus, neither the force nor the displacements used in the aeroelastic analysis are those used in the structural analysis. This "interface" problem must be solved when the finite element structural analysis methods are used to define the elastic properties of the structure.

When the aerodynamic surfaces of the airplane are of reasonably high aspect ratio, say 6 or larger, the interface problem discussed above may be resolved by treating the aerodynamic surfaces as beams. The actual structure is idealized as a beam lying along the elastic axis (or locus of centers of flexure, ref. 8) of the actual structure. Torsional stiffness of the elastic axis is defined as the  $GJ$  distribution. The flexural stiffnesses of the elastic axis are defined as the  $EI_x$  and  $EI_y$  distributions. The volume elements,  $V_i$ , of the lumped parameter description are attached to the elastic axis by rigid, massless members. The structure appears as in the following sketch of a wing.



The lumped masses are located at the centroids and are connected to the elastic axis by the rigid, massless members shown as dashed lines.

The stiffness matrix may be developed for this idealized structure by the finite-element, displacement method just described. The flexibility matrix obtained will be appropriate for the lumped parameter formulation,  $[C_0]$ . No interface problems occur and slopes associated with the lumped masses are readily computed.

**5.2.3 Internal equilibrium.**— The basis for writing internal equilibrium equations for the airplane was introduced in par. 5.2.2. There it was noted that the structural deflections for the lumped parameter formulation are represented by

$$\textcircled{\text{S3}} \quad \{d\} - [\Phi] \{B\} = - [C_o] \left( \{R\} - \{F\} \right) \quad (52)$$

The matrix  $\{R\}$  is the matrix of body force components acting on the lumped masses.

The body force vector on the  $i^{\text{th}}$  lumped mass is due to acceleration and gravity. It is expressed as

$$\vec{R}_i = \int_{V_i} \left( \frac{d^2 \vec{r}'}{dt^2} + \vec{g} \right) \rho_A dV$$

Carrying out the integration results in

$$\vec{R}_i = \left( \frac{d^2 \vec{r}_i'}{dt^2} + \vec{g} \right) m_i \quad (53)$$

where  $d^2 \vec{r}_i' / dt^2$  is the mean acceleration of the centroid of the  $i^{\text{th}}$  lumped mass relative to the earth-fixed-axis system,  $x', y', z'$ . Letting the angular velocity of the airplane about its center of gravity be represented by the vector  $\vec{\omega}$ , the acceleration of the  $i^{\text{th}}$  lumped mass may be expanded to give

$$\begin{aligned} \frac{d^2 \vec{r}_i'}{dt^2} &= \frac{\partial \vec{V}_c}{\partial t} + \vec{\omega} \times \vec{V}_c + \frac{\partial^2 \vec{d}_i}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{d}_i}{\partial t} \\ &\quad + \frac{\partial \vec{\omega}}{\partial t} \times \vec{r}_i + \vec{\omega} \times (\vec{\omega} \times \vec{r}_i) \end{aligned} \quad (54)$$

where

$\frac{\partial}{\partial t}$  is the time rate of change apparent to an observer rotating with the airplane

$\vec{V}_c = \frac{d\vec{r}_{O'}}{dt}$  is the velocity of the airplane center of gravity relative to the earth-fixed-axis system,  $x', y', z'$ .

It is assumed that the rotation rate of large, flexible airplanes is sufficiently small that the Coriolis forces and a portion of the centrifugal forces may be neglected by comparison with the other inertial forces, i.e.,

$$2\vec{\omega} \times \frac{\partial \vec{d}_i}{\partial t} m_i \approx 0 \quad \text{and} \quad \vec{\omega} \times (\vec{\omega} \times \vec{r}_i) \approx 0 \quad (55) \quad (D4)$$

so that the inertial force on the  $i$ th lumped mass is approximately

$$m_i \frac{d^2 \vec{r}_i'}{dt^2} = m_i \left[ \frac{\partial \vec{V}_c}{\partial t} + \vec{\omega} \times \vec{V}_c + \frac{\partial \vec{\omega}}{\partial t} \times \vec{r}_i \right] + m_i \frac{\partial^2 \vec{d}_i}{\partial t^2} \quad (56)$$

In the case of steady reference motion of the airplane, as in par. 5.1.2, the inertial forces acting on the mass elements reduce to

$$m_i \left( \frac{d^2 \vec{r}_i'}{dt^2} \right) = m_i \vec{\omega}_1 \times \vec{V}_{c1} \quad (57)$$

where, again, the subscript 1 denotes the values of the motion variables in the steady reference motion. Also, as in par. 5.1.2, the gravity force on the lumped masses is

$$\vec{g}_1 m_i = m_i g (-\sin \Theta_1 \vec{i} + \cos \Theta_1 \sin \Phi_1 \vec{j} + \cos \Theta_1 \cos \Phi_1 \vec{k}) \quad (58)$$

The perturbation equations of motion were obtained in the case of the rigid airplane in par. 5.1.3. It is shown in app. A, par. 6.3, that the perturbation equations of motion for the

rigid airplane may be expressed in the lumped parameters as

$$[M] \left( \left\{ \dot{v}_p \right\} + [M_1] \left\{ v_p \right\} + [M_2] \left\{ r'_{op} \right\} \right) = [\bar{\phi}]^T \left\{ F \right\} \quad (59)$$

where  $\{v_p\} = \{V\} - \{V_1\}$  is the matrix of perturbation velocities corresponding to the velocity matrix of equation (33). The perturbation matrix  $\{r'_{op}\}$  represents perturbation of the position and orientation of the airplane's center of gravity,

$$\left\{ r'_{op} \right\}^T = [x'_{op}, y'_{op}, z'_{op}, \phi_p, \theta_p, \psi_p] \quad (60)$$

The matrices  $[M_1]$  and  $[M_2]$  are defined in app. A by equations (6.119) for small perturbations and equations (6.120) for large perturbations. The matrix  $[M_1]$  contains coefficients arising from centrifugal accelerations and  $[M_2]$  contains coefficients appropriate to gravity forces.

The perturbation body forces acting on the lumped masses of the airplane are given by

$$\left\{ R \right\} = [m] \left\{ \ddot{d}_p + [\phi] \left\{ \dot{v}_p \right\} + [M_1] \left\{ v_p \right\} + [M_2] \left\{ r'_{op} \right\} \right\} \quad (61)$$

so that the perturbation form for internal equilibrium, equation (52), may be written as

$$\left\{ d_p \right\} - [\bar{\phi}] \left\{ B_p \right\} = - [C_o] [m] \left\{ \ddot{d}_p + [\bar{\phi}] \left\{ \bar{R} \right\} \right\} + [C_o] \left\{ F \right\} \quad (62a)$$

where

$$\left\{ \bar{R} \right\} \equiv \left\{ \dot{v}_p \right\} + [M_1] \left\{ v_p \right\} + [M_2] \left\{ r'_{op} \right\} \quad (62b)$$

This result and equation (59) constitute the perturbation equations of motion for a flexible airplane.

Internal equilibrium for the airplane in reference motion may be expressed as

$$\{d_1\} - [\bar{\phi}] \{B_1\} = - [C_o] \{[m] [\bar{\phi}] (\{a_1\} + \{g_1\}) - \{F_1\}\} \quad (63)$$

where

$[\bar{\phi}] \{a_1\}$  are the lumped mass accelerations in the reference motion and

$[\bar{\phi}] \{g_1\}$  are the reference gravity vector components.

A more convenient formulation of internal equilibrium may be achieved by introducing new variables for the elastic displacement  $\{d\}$ . Let

$$\{d\} = [\phi] \{u\} \quad (64) \quad (S4)$$

where the matrix  $[\bar{\phi}]$  is a transformation matrix (ref. 3, Art. 9.3) and the matrix  $\{u\}$  is a column matrix of generalized coordinates.

It is desirable to require that the generalized coordinates be linearly independent. This is not the case with the elastic displacements  $\{d\}$ , which are displacements relative to the airplane's center of gravity. As such, from app. A equation (6.130), it follows that

$$[\bar{\phi}]^T [m] \{\ddot{d}\} = 0 \quad (65) \quad (S9)$$

There are  $3n$  components of  $\{d\}$  and, since equation (65) represents six linear relations, there are  $3n-6$  linearly independent generalized coordinates  $\{u\}$ . The transformation matrix  $[\phi]$  has  $3n$  rows and  $3n-6$  columns.

A particularly convenient choice for the transformation  $[\phi]$  consists of letting the columns of this matrix be the eigenvectors generated from an eigenvalue problem constructed from the internal equilibrium equations.

Let the airplane be at rest, or at most in uniform translational motion in free space, and let the airplane be in a state of vibration. Internal equilibrium, equation (52), is then represented by

$$\{d\} - [\bar{\phi}] \{B\} = - [C_o] [m] \{\ddot{d}\} \quad (66)$$

which, on utilizing equation (65) to determine the elements of  $\{B\}$ , may be written as

$$\{d\} = - [\bar{C}] [m] \{d\} \quad ; \quad (67a)$$

where

$$[\bar{C}] \equiv \left[ [I] - [\bar{\phi}] [M]^{-1} [\bar{\phi}]^T [m] \right] [C_o] \quad (67b)$$

Introducing the transformation, equation (64), into equation (67a) results in

$$[\bar{\phi}] \{u\} = - [\bar{C}] [m] \{\bar{\phi}\} \ddot{u} \quad (68)$$

where it has been assumed that the transformation matrix is not a function of time. Each column of the transformation matrix may be taken to be a solution to

$$\{\bar{\phi}\}_i u_i = - [\bar{C}] [m] \{\bar{\phi}\}_i \ddot{u}_i$$

which, by a separation of variables (ref. 9, p. 430), may be written as

$$\left[ [I] - \omega_i^2 [\bar{C}] [m] \right] \{\bar{\phi}\}_i = 0 \quad (69a)$$

and

$$\ddot{u}_i + \omega_i^2 u_i = 0 \quad (69b)$$

where  $\omega_i^2$  is the separation constant.

Equation (69a) represents the eigenvalue problem that was being sought. It has  $3n-6$  linearly independent solutions, eigenvectors,  $\{\bar{\phi}\}_i$  corresponding to  $3n-6$  eigenvalues  $\omega_i^2$ . The eigenvectors represent free vibration mode shapes for the airplane. From equation (69b) it is readily apparent that  $\omega_i$  is the frequency of a simple harmonic motion. This is a natural frequency of the freely vibrating airplane. The eigenvectors  $\{\bar{\phi}\}_i$  are the mode shapes, i.e., the shapes into which the structure deforms when vibrating freely.



The transformation of coordinates, equation (64), may now be given more meaning. The displacements  $\{d\}$  that describe the elastically deformed shape of the airplane are a linear combination

$$\{d\} = \sum_{i=1}^{3n-6} \{\phi_i\} u_i$$

of shapes  $\{\phi_i\}$ . The generalized coordinates  $u_i$  are the amplitudes of the shapes. By properly adjusting the amplitudes, the elastic deformation of the airplane under the action of any system of self-equilibrating loads may be represented by equation (64). Thus, the generalized coordinates  $u_i$  are completely satisfactory for expressing elastic motion. A considerable advantage accrues through their use. The internal equilibrium equations, when transformed, assume a form much simpler than that of equation (62).

The perturbation internal equilibrium equations may be written in terms of stiffness as

$$[\bar{K}] \{d_p\} = - [m] \{\ddot{d}_p\} + \{f\} - [m] [\bar{\phi}] \{R\} \quad (70a)$$

where

$$[\bar{K}] \equiv [C_o]^{-1} - [C_o]^{-1} [\bar{\phi}] \left( [\bar{\phi}]^T [C_o]^{-1} [\bar{\phi}] \right)^{-1} [\bar{\phi}]^T [C_o]^{-1} \quad (70b)$$

This result is derived in app. A as equation (6.147). On introducing the transformation and multiplying equation (70) by the transpose of the modal matrix, it is found that

$$[\bar{K}] \{u\} + [\bar{m}] \{\ddot{u}\} = [\phi]^T \{f\} \quad (71)$$

where use has been made of the orthogonality properties

$$[\phi]^T [\bar{K}] [\phi] = [\bar{K}] \quad (72)$$

and

$$[\phi]^T [m] [\phi] = [\bar{m}] \quad (73)$$

from app. A, equations (6.157) and (6.158). In addition, it may be noted that the term corresponding to rigid-body accelerations and gravity  $\{\bar{R}\}$  vanishes. This is a consequence of equation (65), which may be used to write

$$[\phi]^T [\bar{m}] [\bar{\phi}] = 0 \quad (74)$$

This last result may be interpreted as an orthogonality of the free vibration mode shapes  $\{\phi_i\}$ , the columns of  $[\phi]$ , with the columns of  $[\bar{\phi}]$  when weighted by the mass matrix. The columns of  $[\bar{\phi}]$  may be regarded as rigid-body mode shapes, and that term will be applied in the following in calling  $[\bar{\phi}]$  the rigid-body mode shape matrix.

Two obvious advantages have accrued from the transformation. The generalized stiffness and mass matrices,  $[\bar{K}]$  and  $[\bar{m}]$ , are both diagonal matrices. Thus, equation (71) is uncoupled in the left-hand member. If the right-hand member is replaced by a column matrix of generalized perturbation forces, i.e.,

$$\{Q_p\} = [\phi]^T \{f\} \quad (75)$$

internal equilibrium in terms of the generalized coordinates expands into  $3n-6$  scalar equations as

$$\begin{aligned} \bar{m}_1 \ddot{u}_1 + \bar{K}_1 u_1 &= Q_{p1} \\ \bar{m}_2 \ddot{u}_2 + \bar{K}_2 u_2 &= Q_{p2} \\ &\vdots \\ \bar{m}_{3n-6} \ddot{u}_{3n-6} + \bar{K}_{3n-6} u_{3n-6} &= Q_{p3n-6} \end{aligned} \quad (76)$$

When internal equilibrium is expressed by writing out the equations as they are here, the advantage gained from the use of the generalized coordinates  $u_i$  is readily evident. Only one of them occurs in each of the left-hand members. This is the meaning attached to the term "uncoupled." The only coupling in equations (76) is in the right-hand members. The generalized perturbation forces  $Q_{pi}$  are coupled. In general, each is a function of  $u_i$ ,  $\dot{u}_i$ ,  $\ddot{u}_i$  and the motion variables  $u, v, w, p, q, r$  and their time rates of change. That dependence will be discussed later in the subsection regarding aerodynamic derivatives.

It is important to note that the motions governed by the internal equilibrium equations are not motions relative to an inertial reference frame. The elastic displacements  $\{dp\}$  in equation (70) and the generalized displacements  $\{u\}$  in equation (76) describe motion relative to a body axis system. The body axis system may be rotating and accelerating so it is not, in general, an inertial reference frame. Equations (70) and (76) are not exact expressions of Newton's law of mass times acceleration equated to applied force. They are approximations, unless the center of gravity of the airplane is moving with a constant translational velocity. The approximation is valid only if the Coriolis and centrifugal perturbation forces are negligible by comparison with the other forces acting on the airplane as noted by equations (55).

5.2.4 General equations of motion.— In the preceding subsection the lumped parameter formulation was used to formulate internal equilibrium for perturbation motion, equations (62a) and (76). Internal equilibrium for the airplane in its reference motion was given by equation (63). Also, the perturbation equations of motion for motion of the airplane as a whole were obtained by analogy with the rigid airplane perturbation equations of motion. They were given by equation (59).

The system of equations listed is complete for the analysis of the stability of an elastic airplane with two exceptions. The dependence of the aerodynamic and thrust forces on the airplane's motion has not been delineated, and equations of motion for the motion of the airplane as a whole have not been given for the reference motion. Those equations are obtained from general equations of motion. That derivation is precisely the same as the derivation for the rigid airplane. Equations (8) and (9), for reference motion of a rigid airplane, followed directly from the general equations of motion for a rigid airplane, equations (5) and (6).

The only distinguishing feature in the application of these results is obscured at this point of the development. The aerodynamic and thrust forces and moments depend on the airplane's flexibility. If the moments and products of inertia are denoted by the subscript one to indicate that they are evaluated for the airplane shape under the steady reference loads, the reference motion equations of motion may be written immediately as

$$M (Q_1 W_1 - R_1 V_1) = -Mg \sin \Theta_1 + F_{A_{x_1}} + F_{T_{x_1}} \quad (77a)$$

$$M (R_1 U_1 - P_1 W_1) = Mg \cos \Theta_1 \sin \Phi_1 + F_{A_{y_1}} + F_{T_{y_1}} \quad (77b)$$

$$M (P_1 V_1 - Q_1 U_1) = Mg \cos \Theta_1 \cos \Phi_1 + F_{A_{z_1}} + F_{T_{z_1}} \quad (77c)$$

$$-I_{xz_1} Q_1 P_1 + (I_{zz_1} - I_{yy_1}) Q_1 R_1 = M_{A_{x_1}} + M_{T_{x_1}} \quad (77d)$$

$$I_{xz_1} (P_1^2 - R_1^2) + (I_{xx_1} - I_{zz_1}) P_1 R_1 = M_{A_{y_1}} + M_{T_{y_1}} \quad (77e)$$

$$I_{xz_1} Q_1 R_1 + (I_{yy_1} - I_{xx_1}) P_1 Q_1 = M_{A_{z_1}} + M_{T_{z_1}} \quad (77f)$$

and

$$\dot{\Theta}_1 = Q_1 \cos \phi_1 - R_1 \sin \phi_1 \quad (78a)$$

$$\dot{\phi}_1 = P_1 + (Q_1 \sin \phi_1 + R_1 \cos \phi_1) \tan \Theta_1$$

The methods for finding the reference shape of the airplane cannot be described without a detailed description of the aerodynamic and thrust terms. That description is the subject of par. 5.2.5. The problem of computing the reference airplane shape is formulated in par. 5.2.6.

5.2.5 Aerodynamic derivatives.— Up to this point the aerodynamic forces have been introduced simply as a consequence of surface stress, and for the lumped parameter formulation this led to a column matrix of components of mean perturbation aerodynamic forces  $\{f\}$ . The perturbation aerodynamic forces depend on the perturbation motion of the airplane. It is stated in app. A, equation (6.172), and shown in app. B that the dependence may be approximated by the expression:

$$\{f\} = [A_1] \{v_p\} + [A_2] \{\dot{v}_p\} + [A_3] \{d_p\} + [A_4] \{\dot{d}_p\} + [A_5] \{\ddot{d}_p\} \quad (79)$$

As noted in app. B, the coefficient matrices may be determined by wind tunnel measurements. However, theoretical determinations of the matrices in equation (79) are based on inviscid, small perturbation flow theory. Without entering into the detailed derivations of the aerodynamic matrices  $[A_1] \dots [A_5]$ , it is easily shown that equation (79) is an appropriate form.

Solutions to the inviscid flow problem are in the form of a perturbation velocity potential that satisfies an equation of flow and appropriate boundary conditions. The form of the flow equation has no direct impact on the general form of equation (79). The boundary conditions consist of the requirements that the flow be tangent to the surface of the airplane and that certain vortex laws be satisfied in the airplane's wake. As shown in app. B, the boundary conditions at the surface contain only the perturbation potential (not its time rate of change), the perturbations to the airplane's translational and rotational velocity components, and the elastic displacements and their rates of change. The wake boundary condition depends on the frequency of the motion  $\omega$ , but for small frequencies this dependence may be eliminated by an approximation. Thus, the perturbation velocity potential  $\phi$  must have parametric dependence as

$$\phi = \phi(x, y, z, t; \{V_p\}, \{d_p\}, \{\dot{d}_p\}) \quad (80)$$

The perturbation pressure at the surface of the airplane is computed from the perturbation velocity potential using Bernoulli's equation. That computation involves taking the first derivative of  $\phi$  with respect to time. Thus, the pressure force at the  $i^{\text{th}}$  element of the airplane will have the parametric dependence

$$P_i = P_i(\{V_p\}, \{\dot{V}_p\}, \{d_p\}, \{\dot{d}_p\}, \{\ddot{d}_p\}) \quad (81)$$

Finally, the aerodynamic theory is a linear one and the pressure is normal to the surface. The linear dependence in equation (79) is therefore justified and the components of the surface force are proportional to the surface pressure force and the components of a unit vector normal to the surface in the reference airplane shape.

The aerodynamic derivatives for the airplane are computed from equation (79) by using the mode shape matrices. The elastic displacements may be replaced by generalized displacements by using the transformation equation, (64). Thus, the forces and moments acting at the airplane's center of gravity are given by

$$\begin{Bmatrix} f_{A_x} \\ f_{A_y} \\ f_{A_z} \\ m_{A_x} \\ m_{A_y} \\ m_{A_z} \end{Bmatrix} = [\bar{\phi}]^T \left[ [A_1] \{V_p\} + [A_2] \{\dot{V}_p\} + [A_3] [\phi] \{u\} + [A_4] [\phi] \{\dot{u}\} + [A_5] [\phi] \{\ddot{u}\} \right] \quad (82)$$

;

The generalized perturbation aerodynamic forces,  $Q_{p_i}$ , appearing in equation (76) are given by

$$\{Q_p\} = [\phi]^T \left\{ [A_1] \{V_p\} + [A_2] \{\dot{V}_p\} + [A_3] [\phi] \{u\} + [A_4] [\phi] \{\dot{u}\} + [A_5] [\phi] \{\ddot{u}\} \right\} \quad (83)$$

An aerodynamic derivative is the rate of change of a component of force or moment on the airplane due to a change in a motion variable while all other motion variables are held unchanged. Thus, for example,

$$m_{A_y \dot{u}_2}$$

is the aerodynamic derivative relating pitching moment to rate of change of the second generalized elastic coordinate. It is found from equation (82) by setting  $\dot{u}_2$  equal to unity and all other perturbation motion variables to zero. For an airplane represented as  $n$  lumped masses, equations (82) and (83) contain  $(27n^2 - 18n)$  aerodynamic derivatives. If there are 100 lumped masses, then there may be 268 200 aerodynamic derivatives.

Those aerodynamic derivatives which relate forces and moments at the airplane's center of gravity to motion of the airplane as a whole are contained in the first two terms on the right of equation (82). These aerodynamic derivatives correspond to those introduced in the discussion of the rigid airplane and contained in equations (23). This correspondence is delineated in Sec. 7 of this volume and is developed in detail in app. B.

**5.2.6 Airplane reference shape (jig shape determination).**— The selection of the design shape for an airplane is the result of an optimization process involving a large number of design parameters. When an airplane has a significant amount of structural flexibility, allowance must be introduced for the difference in the shape of the airplane as it is when manufactured in fabrication jigs and as it is when subjected to in-flight loads. In the case of a transport airplane the optimization process usually minimizes the drag of the airplane in steady, level, midcruise flight subject to constraints such as minimum body diameters and wing thickness. The result is a design shape that is the optimum shape of the airplane in a design point flight condition. The design shape may differ significantly from the jig shape or the shape of the airplane in off-design point flight conditions. Computation methods for determining the jig and off-design point shape are the subject of this section.

The basic relations required for computation of the off-design point and jig shapes of the airplane have already been developed. Recalling equation (63) and making use of equation (65), it follows that the elastic displacements in the reference flight condition are given by

$$\{d_1\} = -[\bar{C}_1] \left\{ [m_1] [\bar{\Phi}] (\{a_1\} + \{g_1\}) - \{F_1\} \right\} \quad (84)$$

The subscript 1 has been added to the flexibility matrix and the mass matrix to denote that these matrices are dependent upon the reference flight condition mass distribution.

The values of the elements appearing in the acceleration matrix  $\{a_1\}$  and the gravitational matrix  $\{g_1\}$  of equation (84) are specified by the reference flight condition. The aerodynamic and thrust forces  $\{F_1\}$  must be such that the equations of motion for the reference flight condition, equations (75), are satisfied. They may be written in matrix form in an analogy with equation (84) as

$$[\bar{\Phi}]^T [m_1] [\bar{\Phi}] (\{a_1\} + \{g_1\}) = [\bar{\Phi}]^T \{F_1\} \quad (85)$$

where

$$\{F_{A_1}\} + \{F_{T_1}\} = [\bar{\Phi}]^T \{F_1\} \quad (86)$$

as noted by equation (40).

The reference motion of the airplane is a steady motion. In a steady motion the aerodynamic equation, equation (79), reduces to

$$\{f\} = [A_1] \{v_p\} + [A_3] \{d_p\} \quad (87)$$

If it is assumed that the aerodynamic forces are linearly related to the motion variables in the reference motion, then equation (85) may be applied to the reference motion. This is not always appropriate. Thus, let

$$\{F_1\} = [\bar{A}_1] \{V_1\} + [A_3] \{d_1\} + [A_c] \{\delta_1\} + \{F_t\}_1 \quad (88)$$

where a bar has been used on the matrices  $[\bar{A}_1]$  and  $[\bar{A}_2]$  to denote that these matrices may require empirical determination or, if determined from potential flow theory, may have forms that differ from those of their perturbation counterparts. Aerodynamic control surface deflections have been added as the elements of  $\{\delta_1\}$  using an influence coefficient matrix  $[A_c]$  to relate them to aerodynamic forces on the lumped masses. Finally, the thrust forces on the lumped masses have been included as  $\{F_T\}_1$  with the subscript 1 exterior to the brackets to distinguish this matrix from the similar matrix appearing in equation (86).

Equations (84), (85), and (88) may be combined to give

$$[M] (\{a_1\} + \{g_1\}) = [\dot{\Phi}]^T \left\{ [\bar{A}_1] \{V_1\} + [A_c] \{\delta_1\} + \{F_T\}_1 - [A_3] \left[ [I] - [\bar{C}_1] [A_3] \right]^{-1} [\bar{C}_1] \left[ [m_1] [\bar{\Phi}] (\{a_1\} + \{g_1\}) - [\bar{A}_1] \{V_1\} - [A_c] \{\delta_1\} - \{F_T\}_1 \right] \right\} \quad (89)$$

where the definition of the inertia matrix  $[M]$  has been used from equation (35). The control deflections  $\{\delta_1\}$  are the only unspecified quantities appearing in equation (89), with the possible exception of thrust. These quantities must be adjusted so that the six equations represented by equation (89) are satisfied. This balances the airplane into the reference flight condition.



Having balanced the airplane, the deformed shape may be determined. Combining equations (88) and (84) to find

$$\{d_1\} = -\left[[I] - [\bar{C}_1][A_3]\right]^{-1} [\bar{C}_1] \left\{ [m_1][\bar{\phi}] (\{a_2\} + \{g_1\}) - [\bar{A}_1]\{v_1\} - [A_c]\{\delta_1\} - \{F_T\}_1 \right\} \quad (90)$$

all quantities on the right are known from the preceding, so  $\{d_1\}$  is determinable.

The airplane in the jig is not subjected to loads, including those due to gravity. Thus,  $\{d_1\}$  vanishes for the jig shape. Letting the quantities on the right be evaluated for the design flight condition, the displacements  $\{d_1\}$  describe the difference in the design and jig shapes. Off-design point reference motions lead to airplane shapes that are much more readily obtained from the jig shape. The displacements computed from equation (90), when it is evaluated for the off-design conditions, are simply added to the jig shape.

**5.2.7 Perturbation equations of motion.**— The perturbation equations of motion were presented in a matrix formulation in par. 5.2.3. They are given by

$$[M] (\{\dot{V}_p\} + [M_1]\{V_p\} + [M_1]\{r_p\}) = [\bar{\phi}]^T \{f\} \quad (59)$$

and

$$[\bar{K}]\{u\} + [\bar{m}]\{\ddot{u}\} = [\bar{\phi}]^T \{f\} \quad (71)$$

where the aerodynamic contribution to the right-hand members is given by equations (64) and (79) as

$$\{f\} = [A_1]\{V_p\} + [A_2]\{\dot{V}_p\} + [A_3][\bar{\phi}]\{u\} + [A_4][\bar{\phi}]\{\dot{u}\} + [A_5][\bar{\phi}]\{\ddot{u}\} \quad (91)$$

Equations (59) are six in number and equations (71) are  $3n-6$  in number when  $n$  lumped masses are used to represent the airplane. The stability of the airplane is assessed by determining the manner in which the variables  $\{V_p\}$  depend on time. Thus, equations (59) and (71) must be integrated to determine  $\{V_p\}$ . That integration requires the kinematic relations given by equations (20), i.e.,

$$\dot{\Theta}_p = g \cos \Theta_1 - r \sin \phi_1 \quad (20a)$$

$$\dot{\phi}_p = p + (r \sin \phi_1 + q \cos \phi_1) \tan \Theta_1 \quad (20b)$$

The above system of equations constitutes the equations of motion for a completely elastic airplane. Because the number of equations and unknowns is very large, it is of the utmost importance, if at all possible, to reduce the number of equations and unknowns.

One method of simplifying the problem is to use the residual flexibility method. The frequency of the motion in many of the elastic degrees of freedom is very large. When these frequencies are an order of magnitude or more larger than the frequencies of the rigid-body motion, the two motions are only weakly coupled dynamically. However, the elastic deflections due to those elastic degrees of freedom may strongly affect the rigid-body motion. The effect is brought about by quasi-static deflection of the airplane structure. The high frequency of the motion implies that the deflection is very nearly in phase with and in constant proportion to the loads producing it. Thus, generalized inertial forces such as  $\bar{m}_i \ddot{u}_i$  are very small.

Residual flexibility is formulated by partitioning the generalized elastic deflections  $\{u\}$  into two parts, as

$$\{u\} = \begin{Bmatrix} \{u_1\} \\ \{u_2\} \end{Bmatrix}$$

The generalized inertial forces associated with  $\{u_2\}$  are set to zero. Equation (71) is partitioned as

$$[\bar{m}_1] \{u_1\} + [\bar{m}_1] \{\ddot{u}_1\} = [\phi_1]^T \{f\} \quad (92)$$

and

$$[\bar{K}_2]\{u_2\} = [\phi_2]\{f\} \quad (93)$$

This formulation is discussed in considerable detail in app. A. There it is shown that equations (92) and (93) may be combined with equations (59) and (91) so as to eliminate the generalized coordinates  $\{u_2\}$  completely. The result is the set of equations governing the motion of the airplane's center of gravity as

$$\begin{aligned} [M] \left( \{\dot{V}_p\} + [M_1] \{V_p\} + [M_2] \{r'_{op}\} \right) = \\ [\bar{\phi}]^T \left[ [I] - [A_3] [\bar{C}_R] \right]^{-1} \left[ [A_1] \{V_p\} \right. \\ + [A_2] \{\dot{V}_p\} + [A_3] [\phi_1] \{u_1\} - [A_3] [C_R] [m] [\phi_1] \{\ddot{u}_1\} \\ \left. - [A_3] [\bar{C}] [m] [\bar{\phi}] \left( \{\dot{V}_p\} + [M_1] \{V_p\} + [M_2] \{r'_{op}\} \right) \right. \\ \left. + [A_4] [\phi_1] \{\dot{u}_1\} + [A_5] [\phi_1] \{\ddot{u}_1\} \right] \end{aligned} \quad (6.208)$$

and dynamic-elastic equations

$$\begin{aligned} [m_1] \{\ddot{u}_1\} + [\bar{K}_1] \{u_1\} = [\phi_1]^T \left[ [I] - [A_3] [\bar{C}_R] \right]^{-1} \left[ [A_1] \{V_p\} \right. \\ + [A_2] \{\dot{V}_p\} + [A_3] [\phi_1] \{u_1\} - [A_3] [\bar{C}_R] [m] [\phi_1] \{\ddot{u}_1\} \\ \left. - [A_3] [\bar{C}] [m] [\bar{\phi}] \left( \{\dot{V}_p\} + [M_1] \{V_p\} + [M_2] \{r'_{op}\} \right) \right. \\ \left. + [A_4] [\phi_1] \{\dot{u}_1\} + [A_5] [\phi_1] \{\ddot{u}_1\} \right] \end{aligned} \quad (6.209)$$

where the matrix  $[\bar{C}_R]$  is the residual flexibility matrix. The residual flexibility is given by

$$[\bar{C}_R] = [\bar{C}] - [\phi_1] [K_1]^{-1} [\phi_1]^T \quad (6.203)$$

It may be noted that  $[\bar{K}_1]^{-1}$  is the flexibility associated with the dynamically included generalized coordinates  $\{u_1\}$ . Thus,  $[\bar{C}_R]$  is the excess of flexibility of  $[\bar{C}]$  over the flexibility included in the dynamically included portion of the structural behavior. The term "residual flexibility" is appropriate.

The formulation presented by equations (6.208) and (6.209) has not been widely used. An alternate formulation used in some of the evaluations presented in this report has had prior use. The residual flexibility matrix is also given by

$$\left( [\phi_2] [\bar{K}_2]^{-1} [\phi_2]^T - [\bar{C}_R] \right) \left( -[\bar{m}] \{\ddot{d}_p\} + \{f\} \right) = -[\bar{C}] [\bar{m}] [\bar{\phi}] \{R\} \quad (94)$$

$$-[\phi_1] [\bar{K}_1]^{-1} [\phi_1]^T \left( -[\bar{m}] \{\ddot{d}_p\} + \{f\} \right)$$

The alternate formulation follows by substituting equation (94) into equations (6.208) and (6.209). This particular formulation has the disadvantage that all free vibration mode shapes must be computed with high accuracy. That is not always practical. The result is a less accurate evaluation of the residual flexibility than is available with the form given by equation (6.203).

If we introduce generalized coordinates  $q_i$  with the first six of these set into correspondence with the elements of the matrix  $\{r'_{op}\}$ , which appears in the matrix formulation of the equations of motion, then to the degree of approximation appropriate to the small perturbation equations of motion

and 
$$\{v_p\}^T = [\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, \dot{q}_5, \dot{q}_6]$$

$$\frac{\partial}{\partial t} \{v_p\}^T = [\ddot{q}_1, \ddot{q}_2, \ddot{q}_3, \ddot{q}_4, \ddot{q}_5, \ddot{q}_6]$$

Also, letting

$$\{u\}^T = [q_7, q_8, \dots, q_j]$$

the equations of motion given by equations (6.208) and (6.209) of app. A may be merged into a single matrix equation. This equation represents both rigid-body and internal motion for a residual-flexibility formulation and is given by

$$[a] \{\ddot{q}\} + [b] \{\dot{q}\} + [c] \{q\} = \{0\} \quad (95)$$

where

$$[a] \equiv \begin{bmatrix} \left( [M] - [\bar{\phi}]^T [B_1]^{-1} [A_2] \right) \left( -[\bar{\phi}]^T [B_1]^{-1} [A_5] [\phi_1] \right) \\ \left( -[\phi_1]^T [B_1]^{-1} [A_2] \right) \left( [\bar{K}_1] - [\phi_1]^T [B_1]^{-1} [A_5] [\phi_1] \right) \end{bmatrix}$$

$$[b] \equiv \begin{bmatrix} \left( [M] [M_1] - [\bar{\phi}]^T [B_1]^{-1} [A_1] \right) \left( -[\bar{\phi}]^T [B_1]^{-1} [A_4] [\phi_1] \right) \\ \left( -[\phi_1]^T [B_1]^{-1} [A_1] \right) \left( -[\phi_1]^T [B_1]^{-1} [A_4] [\phi_1] \right) \end{bmatrix}$$

$$[c] \equiv \begin{bmatrix} \left( [M] [M_2] \right) \left( -[\bar{\phi}]^T [B_1]^{-1} [A_4] [\phi_1'] \right) \\ \left( [0] \right) \left( [\bar{K}_1] - [\phi_1]^T [B_1]^{-1} [A_3] [\phi_1] \right) \end{bmatrix}$$

and  $[B_1] \equiv [I] - [A_3] \left( [\bar{C}] - [\phi_1] [\bar{K}_1]^{-1} [\phi_1]^T \right)$

By letting  $[\bar{K}_1]$  vanish this set of equations reduces to the equations of motion for an equivalent elastic airplane. If all of the elastic degrees of freedom are included as dynamically participating, i.e., completely elastic airplane, then  $[B_1] = [I]$  and these equations continue to hold. It should be noted the thrust perturbation terms have been deleted from equations (6.208) and (6.209) of app. A.

When the structural dynamics are neglected entirely, only equations of motion governing airplane center-of-gravity motion remain. These are called the equations of motion of an equivalent elastic airplane. The resulting expressions are found by replacing  $[\bar{C}_R]$  by  $[\bar{C}]$  in equation (6.208) and setting terms involving  $u_1$  to zero. The result is

$$\begin{aligned} M(\{\dot{V}_P\} + [M_1] \{V_P\} + [M_2] \{r_{Op}'\}) &= [\bar{\phi}]^T [I] - [A_3] [\bar{C}]^{-1} [A_1] \{V_P\} \\ + [A_2] \{\dot{V}_P\} - [A_3] [\bar{C}] [n_1] [\bar{\phi}] (\{\dot{V}_P\} + [M_1] \{V_P\} + [M_2] \{r_{Op}'\}) \end{aligned} \quad (96)$$

Some aspects of these various formulations require emphasis. The perturbation structural deflections for the equivalent elastic airplane are precisely in phase with the externally applied loads. This is not the case in the completely elastic formulation or in the residual flexibility formulation. The maximum deflection, in general, does not occur at the instant of time when the external loads are a maximum. Also, the deflections may exceed the equivalent static-elastic deflections. This is of considerable importance if the structural motion is strongly coupled with the center-of-gravity motion of the airplane. When it is not, the problem becomes the concern of the flutter engineer.

It is of interest to note that the terms on the right of equation (6.208) consisting of

$$[\bar{\Phi}]^T \left[ [I] - [A_3] [\bar{C}_R] \right]^{-1} \left[ [A_1] \{V_p\} + [A_2] \{\dot{V}_p\} \right]$$

contain the stability derivatives that relate change in the forces and moments at the airplane's center of gravity to changes in the airplane's motion as a whole. They are equivalent to the stability derivatives appearing in equations (23). If the residual-flexibility matrix  $[\bar{C}_R]$  is set to zero, these terms contain the rigid airplane stability derivatives. If  $[\bar{C}_R]$  is set equal to  $[\bar{C}]$ , the total airplane flexibility, so that equations (96) result, they are the stability derivatives for an equivalent elastic airplane. In that case the terms in equation (96) consisting of

$$- [\bar{\Phi}]^T \left[ [I] - [A_3] [\bar{C}_R] \right]^{-1} [A_3] [\bar{C}_R] \{m\} [\bar{\Phi}] \left( \{\dot{V}_p\} + [M_1] \{V_p\} \right)$$

may be included into the equivalent elastic stability derivatives. This is sometimes referred to as having included inertial relief. The terminology stems from the practice of constructing equivalent elastic stability derivatives using static-elastic representations of the airplane.

## 6. STABILITY CRITERIA

Static and dynamic stability criteria are presented and discussed in this section. The problem of establishing handling-qualities criteria is beyond the scope of this investigation. However, physical interpretations of stability criteria and connections with previously established handling qualities are pointed out.

### 6.1 Static Stability Criteria

Static stability is defined as the tendency of an airplane to develop forces or moments that directly oppose an instantaneous disturbance of a motion variable from a steady-state (i.e., equilibrium or trim state) flight condition. For example, when the nose of an airplane is raised relative to the flight path and, as a consequence, the airplane develops a nose-down moment, the airplane is said to be statically stable for such a disturbance.

A static stability criterion is defined as a rule by which steady-state flight conditions may be categorized as stable, neutrally stable, or unstable.

In other contexts, the words "static stability criterion" have been used as a requirement for an arbitrary minimum static margin. For example, the military specification for flying qualities (ref. 10, par. 3.3.1.1) requires a negative value of  $C_{m_\alpha}$  at all times, which implies a positive static margin.

In still another interpretation, the Civil Airworthiness Requirements (ref. 11, art. 4b.151-155) associate stability criteria with stick-force versus speed behavior.

The reasons for presenting and defining static stability criteria as given are as follows.

1. The definitions of stable, unstable, and neutrally stable given are clear so that judgment and opinions are eliminated as factors.
2. These definitions lead directly to important aerodynamic derivatives and show how they are related to static stability behavior.

The reader will notice that these definitions are largely independent of notions of stability and stability criteria associated with control force or control surface displacement. Specifically, this report does not deal with:

1. Stick-free stability;
2. Stability as affected by the feel system including bob-weights;
3. Stability augmentation systems in general.

D10

It is recognized that when control surfaces are allowed to float, or when springs or other devices are added, the longitudinal stability derivatives and associated control

characteristics can be significantly affected. Such effects are not discussed directly in this report. However, discussions of the effects of the derivatives do apply to those cases.

The criteria for static stability are summarized in table 1. These criteria are equally valid for rigid and elastic airplanes. It should be noted that they are presented as expressions of local slope behavior. For that reason they apply (as a local linearization) to situations where aerodynamic forces behave in a nonlinear manner. This is important because airplanes do behave in a nonlinear fashion in many instances, for example in stall and pitch-up.

It should also be noted that, although criteria of table 1 evolve from the definition of static stability criteria used here, they vary considerably in importance. For example,  $\partial M_y / \partial \alpha (\sim C_{m_\alpha})$  is of much greater practical importance than  $\partial F_y / \partial v (\sim C_{y_\beta})$ . This will be discussed in more detail in Sec. 7. Notice also from table 1 that under the adopted definition of static stability, the partials,  $\partial M_x / \partial u (\sim C_{m_u})$  and  $\partial M_y / \partial v (\sim C_{l_\beta})$ , do not belong. This implies that for static stability under the current definition, the signs of  $C_{m_u}$  and  $C_{l_\beta}$  are not important. However, in the practical case these derivatives are important. An unusual feature of table 1 is that it includes moment derivatives with respect to rotational velocities. Such derivatives are normally associated with dynamic stability and not with static stability. The reason for their appearance must be found in the definition of static stability. The physical justification for including these moment derivatives in static stability considerations is that steady-state flight can actually involve constant rotational velocities.

The physical meanings of the criteria are stated below.

Criterion
An airplane is statically stable for a forward speed disturbance, $u$ , if: $\partial F_x / \partial u < 0$ (97)

The physical meaning of this criterion is that as a consequence of an increase in forward speed,  $u$  (along the  $x$ -axis), a force must be generated that tends to oppose the increase in speed.

Criterion
An airplane is statically stable for a side speed disturbance, $v$ , if: $\partial F_y / \partial v < 0$ (98)



TABLE 1. – SUMMARY OF STATIC STABILITY CRITERIA

General form of static stability criterion	Approximate or alternate form	Importance to handling qualities	Reference 10 MIL-F-8785
$\frac{\partial F_x}{\partial u} < 0$	$C_{x_{su}} < 0$ or $C_{D_u} > 0$ (No thrust effect)	Needed for stable phugoid. Not important if throttle response is good.	Par. 3.3.6 limits the phugoid divergence. No direct requirement.
$\frac{\partial F_y}{\partial v} < 0$	$C_{y_\beta} < 0$ (No thrust effect)	Helps pilot in perceiving the sideslip. Allows skidding turns at low altitude (wings level).	Pars. 3.4.3 and 3.4.8 interpreted to mean $C_{y_\beta} < 0$
$\frac{\partial F_z}{\partial w} < 0$	$C_{L_\alpha} > 0$	Primary means for flight path control. Significant to short period. Always satisfied before stall.	Par. 3.3.3* specifies short period requirements. No direct requirement.
$\frac{\partial M_z}{\partial \beta} > 0$	$C_{n_\beta} > 0$	Needed to maintain straight flight path.	Pars. 3.4.3, 3.4.4, and 3.4.5 interpreted to mean $C_{n_\beta} > 0$
$\frac{\partial M_y}{\partial \alpha} < 0$	$C_{m_\alpha} < 0$	Affects time history of pitch response.	Par. 3.3.1 interpreted to mean $C_{m_\alpha} < 0$
$\frac{\partial M_x}{\partial p} < 0$	$C_{l_p} < 0$	Affects time history of roll response. Affects Dutch roll damping.	Par. 3.4.1* specifies Dutch roll requirement. Par. 3.4.16 specifies roll performance.
$\frac{\partial M_y}{\partial q} < 0$	$C_{m_q} < 0$	Affects damping of short period (increases pitch stiffness).	Par. 3.3.5* specifies short period requirements. No direct requirements.
$\frac{\partial M_z}{\partial r} < 0$	$C_{n_r} < 0$	Affects Dutch roll damping (increases yaw stiffness).	Par. 3.4.1* specifies Dutch roll requirements. No direct requirements.
$\frac{\partial M_y}{\partial u} > 0$	$C_{m_u} > 0$	Improves speed control. Provides warning of inadvertent over or under speed. Affects stick-force behavior.	No direct requirement, but par. 3.3.3 implies that violation is allowed transonically.
$\frac{\partial M_x}{\partial v} < 0$	$C_{l_\beta} < 0$	Warns pilot of existence of sideslip. Allows emergency roll control. Affects Dutch roll.	Pars. 3.4.3, 3.4.6, and 3.4.7 interpreted to mean $C_{l_\beta} < 0$

\*MIL-F-8785 recognizes augmentation-on and -off cases. This document deals only with unaugmented cases.

TABLE 1.— SUMMARY OF STATIC STABILITY CRITERIA (CONCLUDED)

General form of static stability criterion	Approximate or alternate form	Reference 44 FAR - Part 25	Reference 11 British CAR, section D
$\frac{\partial F_x}{\partial u} < 0$	$C_{x_{su}} < 0$ or $C_{D_u} > 0$ (No thrust effect)	No direct requirement	No direct requirement
$\frac{\partial F_y}{\partial v} < 0$	$C_{y_\beta} < 0$ (No thrust effect)	Par. 25.177(c) interpreted to mean $C_{y_\beta} < 0$	Par. 7.3 interpreted to mean $C_{y_\beta} < 0$
$\frac{\partial F_z}{\partial w} < 0$	$C_{L_\alpha} > 0$	No direct requirement	No direct requirement
$\frac{\partial M_z}{\partial \beta} > 0$	$C_{n_\beta} > 0$	Par. 25.17(a) interpreted to mean $C_{n_\beta} > 0$	Par. 7.2 interpreted to mean $C_{n_\beta} > 0$
$\frac{\partial M_y}{\partial \alpha} < 0$	$C_{m_\alpha} < 0$	No direct requirement but Pars. 25.173 & 25.175 interpreted to mean $C_{m_\alpha} < 0$	Par. 2.1 requires $\frac{dC_m}{dC_L} < +.05$
$\frac{\partial M_x}{\partial p} < 0$	$C_{l_p} < 0$	No direct requirement	No direct requirement
$\frac{\partial M_y}{\partial q} < 0$	$C_{m_q} < 0$	No direct requirement, but Par. 25.181 requires all short periods to be heavily damped	No direct requirement, but Par. 8.1 requires all short periods to be heavily damped
$\frac{\partial M_z}{\partial r} < 0$	$C_{n_r} < 0$	No direct requirement, but Par. 25.181 requires all short periods to be heavily damped	No direct requirement, but Par. 8.1 requires all short periods to be heavily damped
$\frac{\partial M_y}{\partial u} > 0$	$C_{m_u} > 0$	Par. 25.175 (c) implies that violation is not allowed	Par. 31.2 implies that violation is not allowed
$\frac{\partial M_x}{\partial v} < 0$	$C_{l_\beta} < 0$	Par. 25.177 (b) interpreted to mean $C_{l_\beta} < 0$	Par. 7.1 interpreted to mean $C_{l_\beta} < 0$

The physical meaning of this criterion is that as a consequence of a side speed disturbance,  $v$  (along the  $y$ -axis), a force is generated that tends to oppose  $v$ . The approximation,  $v \approx \beta V_{c1}$  will be used.

Criterion

An airplane is statically stable for a vertical speed disturbance,  $w$ , if:  $\partial F_z / \partial w < 0$  (99)

The physical meaning of this criterion is that as a consequence of a positive velocity disturbance,  $w$  (along the  $z$ -axis), a force is generated that tends to oppose  $w$ . The approximation  $\dot{w} \approx \alpha V_{c1}$  will be used.

Criterion

An airplane is statically (directionally) stable for a sideslip disturbance,  $\beta$ , if:  $\partial M_z / \partial \beta > 0$  (100)

The physical meaning of this criterion is that as a result of an angle of sideslip disturbance,  $\beta$ , the airplane weathercocks into the new relative wind.

Criterion

An airplane is statically (longitudinally) stable for an angle-of-attack disturbance,  $\alpha$ , if:  $\partial M_y / \partial \alpha < 0$  (101)

The physical significance of this criterion is that as a result of an angle-of-attack disturbance,  $\alpha$ , the airplane weathercocks into the new relative wind.

Criterion

An airplane is statically stable for a disturbance in roll velocity,  $p$ , if:  $\partial M_x / \partial p < 0$  (102)

The physical meaning of this criterion is that as a result of an increase in rolling velocity,  $p$ , a moment is generated that tends to oppose the increase in rolling velocity.

Criterion

An airplane is statically stable for a disturbance in pitching velocity  $q$ , if:  
 $\partial M_y / \partial q < 0$  (103)

The physical meaning of this criterion is that as a result of an increase in pitching velocity,  $q$ , a moment is generated that tends to oppose the increase in pitching velocity.

Criterion

An airplane is statically stable for a disturbance in yawing velocity  $r$ , if:  
 $\partial M_z / \partial r < 0$  (104)

The physical meaning of this criterion is that as a result of an increase in yawing velocity,  $r$ , a moment is generated that tends to oppose the increase in yawing velocity.

Under the definition of static stability used in this report, the partial differential  $\partial M_y / \partial u$  ( $\sim C_{m_u}$ ) does not qualify as a static stability parameter. However, as will be shown,  $C_{m_u}$  has important consequences to longitudinal stability from the viewpoint of the pilot. In addition, in much of the literature, this parameter is identified with longitudinal stability.

It is noted that a positive sign of  $\partial M_y / \partial u$  means physically that as a result of an increase in forward speed, the airplane noses up, which would tend to slow the airplane down because of the resulting drag increase plus the increase in gravitational pull along the body x-axis. Therefore, an airplane would have stable pitch moment versus speed behavior if:  $\partial M_y / \partial u > 0$ .

Under the definition of static stability used in this report, the partial differential  $\partial M_x / \partial v$  ( $\sim C_{l\beta}$ ) does not qualify as a static stability parameter. Nevertheless, this derivative has an important effect on stability and on handling qualities.

The physical significance of this is that for a positive sideslip disturbance (nose left) the airplane tends to roll away from the disturbance, i.e., roll to the left. If the airplane rolls about its stability x-axis as a result of this, it is easily seen that this tends to diminish the effective sideslip angle. For this reason, some investigators identify  $C_{l\beta}$  as a lateral stability parameter even though strictly speaking the derivative should not be considered as such.

## 6.2 Dynamic Stability Criteria

Dynamic stability is defined as the tendency of the amplitudes of the perturbed motion of an airplane to decrease to zero or to values corresponding to a new steady state at some time after the disturbance has stopped. For example, consider an airplane disturbed in pitch from a steady-state flight condition. If the resulting perturbed motion is damped out after some time, the airplane's motion becomes steady. If the new state is not significantly different from the original one, the airplane is called dynamically stable. The subject of dynamic stability, then, deals with the behavior of the perturbed motion of an airplane about some steady-state flight path.

A dynamic stability criterion is defined as a rule by which perturbed motions are categorized as stable, neutrally stable, or unstable.

In other contexts a dynamic stability criterion has been interpreted as a requirement for specific response characteristics or for meeting specific frequency-damping relations. This type of interpretation is embodied in the military specification for flying qualities (ref. 10) and its proposed revision as documented in ref. 12. There are important connections between dynamic stability criteria (viewed as mathematical statements of stability) and the handling-qualities criteria of refs. 10 and 12. Therefore, where the need for physical

interpretation of the stability criteria is established in this report, the connections with handling qualities are pointed out.

Dynamic stability criteria are established covering the linear and nonlinear equations of motion of an airplane. These criteria apply to rigid, equivalent elastic, and completely elastic descriptions of airplanes provided the corresponding equations of motion are written in the form pertinent to the criteria. Table 2 presents a summary of dynamic stability criteria and their relationships to the various forms of the equations of motion. The arrangement of the equations of motion into the required forms was discussed in Sec. 5. Those combinations of criteria and equations which are most commonly used in airplane stability analysis are identified in table 2 with heavy lines.

**6.2.1 Characteristic equation methods.**— When airplane dynamic behavior can be approximated by assuming that motion perturbations relative to a steady state are small, it is possible to reduce the equations of motion to a set of linear, second-order, ordinary differential equations with constant coefficients. It is shown in app. A that in such a case these equations can be reduced to the following general form:

$$\{\dot{\mathbf{x}}\} = [\mathbf{A}]\{\mathbf{x}\} \quad (105)$$

Taking the Laplace transformation and rearranging equation (105), it follows that

$$[\mathbf{s}\mathbf{I} - [\mathbf{A}]]\{\mathbf{x}_s(s)\} = \{\mathbf{x}_t(t_0^+)\} \quad (106)$$

where:

1.  $\mathbf{s} = \sigma \pm j\omega$  = complex frequency variable
2. The subscripts  $s$  and  $t$  are used to distinguish between the functional relationships  $\{\mathbf{x}_s\}$  and  $\{\mathbf{x}_t\}$

The quantities  $\sigma_j$  and  $\omega_j$  are, respectively, the real and the imaginary parts of the roots  $s_j$  of the characteristic equation:

$$\|\mathbf{s}\mathbf{I} - [\mathbf{A}]\| = 0 \quad (107)$$

TABLE 2.- SUMMARY OF DYNAMIC STABILITY CRITERIA

Stability criteria	Equation type				
	Small perturbations		Large perturbations	Arbitrary perturbations	Arbitrary motions
	Linear	Nonlinear	Nonlinear	Nonlinear	Nonlinear
Characteristic equation methods criteria	Applicable	Not applicable	Not applicable	Not applicable	Not applicable
Time histories criterion	Applicable	Applicable	Applicable	Applicable	Applicable
Energy decay methods criteria	*	Applicable	Applicable Research required	Applicable Research required	Applicable Research required
Lyapunov theory Criterion	*	Applicable Research required	Applicable Research required	Applicable Research required	Not applicable

\* Additional research required for nonautonomous systems

Equation (107) follows from the condition that nontrivial solutions  $\{X_s(S)\} \neq 0$  to equation (106) for  $\{X_t(t_0^+)\} = 0$  are allowed only if equation (107) is satisfied.

The following dynamic stability criterion can then be formulated.

**Criterion**

If the airplane equations of motion are linear and autonomous, then the airplane behavior is said to be:

- stable, if the real parts of the roots of the characteristic equation are all negative,
- neutrally stable, if there are one or more roots of the characteristic equation with zero real parts and the remaining roots all have negative real parts, and
- unstable, if there is at least one root of the characteristic equation with a positive real part.

(108)

In most of the standard literature (refs. 4 and 13 through 17), dynamic stability of airplanes is treated from this viewpoint.

The possibility exists for  $A$  in equation (105) to have elements that are known functions of time. This occurs in steady climbs and dives when dynamic pressure is allowed to vary. In such a case, equation (105) is still linear but is called nonautonomous, and the equations assume the form:

$$\{\dot{x}\} = [A(t)] \{x\} \quad (109)$$

For this type of equation, no simple stability theory is known. (The more complicated stability theory of Lyapunov, which is discussed in par. 6.2.4 and in app. A, could be used.) However, the writers of this report feel that equation (109) is valid, although no proof has been found justifying the approach. The approach consists of applying the characteristic equation method to equation (109) with the following modifications. The characteristic equation considered is:

$$\|[A(t)] - \lambda [1]\| = 0 \quad (110)$$



The following dynamic stability criterion is postulated.

Criterion

When the real parts of the roots of the characteristic equation, equation (110), are negative for  $t = 0$ , as well as for  $t = t_1$ , where  $t_1$  is the practical limit of the time interval considered, then the airplane is stable in that time interval.

(111)

As stated, this criterion needs proof. The proof may lie in defining the quantity  $E$  as equal to the total kinetic energy of the airplane in the perturbed state and also defining  $\dot{E} = dE/dt$ . In that case,  $E$  takes the place of the Lyapunov function. If criterion (111) is satisfied it can be interpreted to mean, according to Lyapunov's theorem 1, that  $\dot{E}$  is negative at the beginning and end of the time interval.

Whether or not in a practical case the dynamic stability criteria (108) are satisfied can be determined by solving directly for the roots of the characteristic equation. A technique for determining stability behavior from the characteristic equation without solving for the roots is known as Routh's criterion. The reader is cautioned not to confuse this with what is sometimes called the Routh-Hurwitz criterion. Routh and Hurwitz both developed similar, but not identical, criteria. However, from the standpoint of calculations Routh's is the more direct approach and is discussed in app. A.

There are many techniques used in systems, analyses, and synthesis techniques (control theory) that may be applied to the perturbed airplane equations of motion of the form of equation (105). Some of the more widely used, for example Bode diagrams, Nichol's charts, the Nyquist criterion, root locus plots, phase trajectories, etc., can be found in the literature (for example, refs. 18 through 22). Most of these techniques were generated for special types of problems and their use is restricted because of limitations imposed by the number of assumptions and/or the effort required in their application.

Applications of the characteristic equations method are limited to linear differential equations of motion; they cannot be used where significant nonlinearities are encountered. However, when the equations of motion can be linearized, the characteristic equations method represented by criterion (108) is a most efficient technique for determining airplane stability behavior. In addition to determining the stability behavior, the roots of the characteristic equations can be used for other analyses. For example, the frequency and damping characteristics, imaginary and real parts of the roots, are used extensively in

handling-qualities analyses and stability augmentation systems design. Examples of such uses may be found in refs. 4, 10, and 12 through 17.

6.2.2 Time histories.— When the equations of motion of an airplane cannot be linearized, it has been common practice to base judgment of stability behavior on observation of traces of time history solutions of the equations of motion.

The resulting traces are judged to determine stability.

**Criterion**

When the motions of an airplane following a disturbance from steady-state flight are determined by a time history, the behavior is:

- stable, if the motions remain in proximity to the steady state,
- neutrally stable, if the motions are undamped and oscillatory about some steady state,
- unstable, if the motions diverge from the steady state either linearly, exponentially, oscillatorily, or in any combination.

(112)

It should be recognized, however, that for nonlinear equations of motion there may be combinations of locally stable, neutrally stable, or unstable regimes. An example of what can be encountered is airplane pitchup above a certain angle of attack. If motions were investigated for small angles (caused by a small initial disturbance), stable motion would be observed. However, if a large disturbance were introduced which would force the angle of attack above the pitchup angle, divergence would occur. It is easily realized that one stable case does not guarantee airplane stability throughout its operating limits. Practically, this problem is resolved by running several time histories with various disturbance magnitudes.

6.2.3 Energy decay methods.— A relatively new and unknown type of stability analysis is the energy decay method, which is discussed in refs. 23 and 24. The fundamental idea behind energy decay methods is that in dynamically stable systems, energy is being dissipated. In the case of linear differential equations with constant coefficients, it is possible to show the inverse, i.e., that if energy is being dissipated the corresponding system is dynamically stable. Extension to nonlinear equations of motion can be justified by

applying the Lyapunov stability theory discussed in app. A. The approach can be stated in the following steps.

1. Derive expressions for the total perturbed energy,  $E$ , of the airplane.
2. From (1) derive the  $\Delta E$  required to make the airplane appear to be a conservative system in the first half cycle of oscillation, i.e., neutrally stable.
3. Examine the sign of  $\Delta E$ .

The following dynamic stability criterion can then be formulated.

**Criterion**

If:  $\Delta E > 0$ , the airplane is stable.

$\Delta E = 0$ , the airplane is neutrally stable or undisturbed.

$\Delta E < 0$ , the airplane is stable.

(113)

A theoretical discussion on how to apply this criterion is given in ref. 24. However, because of algebraic complexities it is not considered practical to use the above criterion in cases involving nonlinear equations of motion. For these cases, Hahn (ref. 25) suggests an energy decay method based on an idea by Lebedev. This idea was further developed by Roskam (ref. 23). There, stability was connected with energy decay through the parameter:

$$F = \frac{\int_{t_2}^{t_3} T \, dt}{\int_{t_1}^{t_2} T \, dt} \quad (114)$$

where  $T$  is the perturbed kinetic energy,  $t_1$  is the beginning of a time interval during which the motion of the airplane is being studied,  $t_3$  is the end of that time interval, and  $t_2$  is the midpoint of that time interval. The criterion for stability in this case follows.

**Criterion**

If:  $F < 1$ , the airplane is stable.

$F = 1$ , the airplane is neutrally stable.

$F > 1$ , the airplane is unstable.

(115)

It is shown in ref. 23 that the condition  $F < 1$ , indicating stability, is satisfied in the case of stable, linear, small perturbation equations of motion.

The advantage of the latter criterion is that it applies to nonlinear equations of motion. A disadvantage is that either considerable numerical work or a computer program is required.

Because energy decay methods have not been widely applied, their limitations, advantages, and disadvantages have not been assessed, although it is felt that there should be few limitations because of the general nature of the approach. However, for linear, autonomous, small perturbation equations of motion, this approach will probably prove to be less efficient than the characteristic equations method.

6.2.4 Lyapunov's method.— The time history method was suggested as a way to determine the stability behavior of an airplane when the equations of motion are nonlinear. However, with the time history method it is necessary to solve the equations of motion. Lyapunov has devised a stability theory for both linear and nonlinear differential equations of perturbed motion that obviates the necessity to solve these equations.

Lyapunov's stability theory has received little attention from airplane stability and control engineers. For this reason, an introduction to this theory as well as some pertinent definitions and theorems are given in app. A. The potential applications of the analysis techniques devised by Lyapunov, and by others who have followed his approach, are quite numerous because of the generality of the approach. Rather than solving any particular problem, Lyapunov realized that the stability of dynamic systems (including moving bodies) was a problem of studying the behavior of differential equations in general. He devised two classes of approach, one for equations whose solutions are known functions of time and another for equations of motion written in the perturbation form. The first approach, using known solutions, is similar to the use of the stability criterion for time histories.

The second approach, called the "direct" or "second" method of Lyapunov, requires choosing a "Lyapunov function" and relating the behavior of this function to the behavior of the differential equations of motion. Owing to a theorem attributed to Zubov, Lyapunov's direct method becomes a particularly attractive approach to the problem of nonlinear airplane stability behavior. Because of its similarity to the more familiar characteristic equation approach it is felt that Zubov's theorem should appear as a logical extension of that approach. In fact, as shown in app. A, it is possible to prove criterion (108) (stable roots for characteristic equations) using Zubov's theorem for linear,

autonomous equations. However, the application of Zubov's theorem would be more useful for nonlinear equations.

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The large perturbation equations of motion of an airplane can be written in the form:

$$\{\dot{x}\} = [F(\{x\}, t)]\{x\} \quad (116)$$

Also, nonlinear, small perturbation equations with nonlinear, aerodynamic, cross-coupling terms, can be written in this form. Before stating the stability criterion for these nonlinear equations, however, some definitions are required. First, the equation

$$\left\| \frac{1}{2} \left[ [F(\{x_R\}, t_R)]^T + [F(\{x_R\}, t_R)] \right] - \lambda [1] \right\| = 0 \quad (117)$$

will be called the "quasi-characteristic equation" where  $\{x_R\}$  and  $t_R$  are defined as values belonging to a representative set of  $x$  and  $t$ . Next, by a "representative set," the following is meant. For given initial disturbances, the solutions to the equations of motion (116) yield a time sequence of values of the motion variables  $\{x\}$ . In most practical cases the engineer will have an idea of the practical limits of the perturbed motions the airplane can experience. In other words, the engineer can make a reasonable estimate of the "cylindrical neighborhood" surrounding the time axis, within which the motion takes place. Choosing discrete values of  $\{x\}$  and  $t$ , called  $\{x_R\}$  and  $t_R$ , within practical limits related to the steady-state flight condition in accordance with these ideas, generates a representative set of  $\{x\}$  and  $t$ . Analogous to this choice of a representative set is the selection of combinations of Mach numbers, dynamic pressures, angles of attack and angles of sideslip for which wind tunnel data is to be obtained or stability is to be assessed in the usual analytical approach.

Finally, the eigenvalues,  $\lambda$ , that will satisfy equation (117) are called the eigenvalues of the quasi-characteristic equation.

Using the above definitions, the application of Zubov's theorem as a dynamic stability criterion is postulated as follows.

#### Criterion

If the eigenvalues of the quasi-characteristic equation are nonpositive ( $\leq 0$ ) for each  $\{x_R\}$  and  $t_R$  in a representative set of  $\{x\}$  and  $t$ , then the airplane is stable.

(118)

Unlike the other stability criteria presented in this section, this criterion has no neutral or unstable counterparts. It is shown in app. A that Lyapunov's direct method has certain limitations and that the existence of positive eigenvalues does not necessarily imply instability. In other words, criterion (1.18) is necessary but not sufficient.

It is emphasized that the use of Zubov's theorem as a basis for determining stability has limitations and disadvantages. Particularly important is the consideration that proving Zubov's theorem requires the use of a particular Lyapunov function, which may lead to very rough stability analyses. Another disadvantage is loss of physical feel for the problem until familiarity with and understanding of the direct method are achieved. Where the engineer can visualize the predicted motions, the time history approach has a distinct advantage.

The particular method presented here, using Zubov's theorem, has no restrictions on the types of perturbation equations to which it can be applied. However, the criterion only pertains to dynamic stability and, as stated above, there are no neutral or unstable counterparts. Also, it will not always predict stability for stable airplanes. This approach has not been sufficiently explored to adequately assess its value.

## 7. METHODS FOR DETERMINING STABILITY DERIVATIVES

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### 7.1 General Considerations

Aerodynamic derivatives are used to relate changes in the aerodynamic forces and moments to changes in the airplane's attitude, motion, and shape due to elastic deformation and control deflections. Those derivatives associated with airplane translation and rotation are generally called the airplane's stability derivatives. Here, those associated with elastic deformation are also included in that classification.

The stability derivatives must be estimated in order to evaluate the stability characteristics of an airplane configuration. Several methods for obtaining estimates are usually used in combination. These methods divide roughly into three categories: (1) estimates based on numerical solutions to the equations of fluid dynamics, (2) estimates based on semi-empirical handbook data, and (3) estimates based on experimental wind tunnel data. The objective here is to describe the best available analytical techniques. These techniques are then evaluated by comparing computed values for stability derivatives with their values measured in wind tunnel and flight tests and by carefully investigating the approximations involved in the theoretical and handbook methods.

There are 24 stability derivatives associated with the translational and rotational degrees of freedom that are usually regarded to have some importance to the stability characteristics of large, flexible airplanes. They are listed in table 3 along with an evaluation of their relative importance. The degree of confidence that can be placed on their predicted values when wind tunnel measurements are used in conjunction with handbook and theoretical techniques is also shown. This table reflects a consensus existing among experienced stability and control engineers at the time of writing. It illustrates the need for improved techniques for estimating lateral-directional stability derivatives.

The advent of large digital computers has greatly enhanced the theoretical approach to estimating stability derivatives. One conclusion of this study is that digital computer programs based on solutions to the equations of fluid dynamics constitute the best method for estimating stability derivatives. These programs can incorporate empirical corrections in somewhat the same manner as the handbook methods. They can also use data from wind tunnel pressure models and the results of flow visualization studies as empirical corrections in a way that is impossible in the current handbook methods.

The computer programs generally use influence coefficient theory. Aerodynamic influence coefficients give the change in pressure at a small region of the airplane's surface due to a change in surface incidence at that and any other small region of the airplane's surface. Structural influence coefficients give the change in incidence and position at a small region due to a change in force applied to that and any other region.

TABLE 3. – RELATIVE IMPORTANCE AND PREDICTION ACCURACY OF STABILITY DERIVATIVES

Longitudinal			Lateral-directional		
Derivative	Relative importance	Estimated accuracy*	Derivative	Relative importance	Estimated accuracy*
$C_{L\alpha}$	Primary	Good	$C_{n\beta}$	Primary	Good
$C_{m\alpha}$	Primary	Good	$C_{l\beta}$	Primary	Good
$C_{mq}$	Primary	Good	$C_{lp}$	Primary	Good
$C_{mu}$	Primary	Good	$C_{nr}$	Primary	Fair
$C_{m\dot{\alpha}}$	Primary	Fair	$C_{np}$	Primary	Poor
$C_{Du}$	Primary	Good	$C_{lr}$	Primary	Poor
$C_{D\alpha}$	Secondary	Good	$C_{Y\beta}$	Primary	Good
$C_{Lu}$	Secondary	Good	$C_{Yp}$	Secondary	Poor
$C_{L\dot{\alpha}}$	Secondary	Poor	$C_{n\dot{\beta}}$	Secondary	Poor
$C_{Lq}$	Secondary	Good	$C_{Yr}$	Secondary	Fair
$C_{D\dot{\alpha}}$	Minor	Poor	$C_{Y\dot{\beta}}$	Minor	Poor
$C_{Dq}$	Minor	Fair	$C_{l\dot{\beta}}$	Minor	Poor

\*Estimated prediction accuracy assumes use of theoretical, handbook, and wind tunnel data.



The influence coefficients can be used in a formulation of the stability derivatives for a rigid airplane, an equivalent elastic airplane, or for the elastic degrees of freedom of an airplane represented as completely elastic. The aerodynamic influence coefficients are based on either lifting surface theory or lifting line theory, although lifting surface theory is the more general and is not restricted in application to high aspect ratio wings as is lifting line theory.

As noted, the aerodynamic influence coefficients can either be combined with the structural influence coefficients to formulate equivalent elastic stability derivatives or be used by themselves to formulate rigid stability derivatives. The ratio of these two stability derivatives or the difference between them can then be used to correct handbook estimates of rigid stability derivatives for the effects of elasticity. These procedures can also be used to correct derivatives measured in the wind tunnel using rigid or nearly rigid models. It represents the best available method for correcting rigid stability derivatives.

The most widely known source of information for estimating stability derivatives is the USAF Stability and Control Handbook (ref. 6). It was the primary handbook source used in the evaluation. This handbook, as well as all others, utilizes empirical and theoretical data determined over a range of certain parameters associated with airplane configuration, e.g., aspect ratio and tail volume, and with the airplane reference flight condition. The stability derivatives of a particular airplane are found by interpolation of the handbook data.

A serious deficiency in the handbook is that the effects of structural flexibility are either not accounted for or are accounted for only through coarse approximations. The handbook methods are not sufficient in themselves to treat flexible, low aspect ratio wing and tail surfaces.

Tables 4 and 5 summarize the results of the evaluation of methods for calculating stability derivatives. The estimates based on lifting line theory (ref. 26) contain empirical corrections and were made using a computer program that is typical of those available throughout the industry. Those based on lifting surface theory (ref. 27), however, were made without empirical corrections. It is important to note this because the lifting surface technique used in this evaluation was developed for wing optimization studies. Leading edge suction is poorly represented in this program; hence, induced drag and yaw due to roll are poorly estimated. The program will only represent an airplane in symmetric flight or a flat airplane in roll. In addition, a first approximation to derivatives that involve unsteady aerodynamics is not an immediately accessible feature. All of these aspects reflect against the lifting surface estimates, although they may be overcome in future programs.

Tables 6, 7, and 8 summarize the limits of applicability of the techniques for obtaining stability derivative estimates. Tables 6 and 7 show the applicability of techniques using aerodynamic influence coefficients—table 6 concerns techniques based on lifting surface theory and table 7 those based on lifting line theory. The applicability of handbook methods is shown in table 8.

TABLE 4. - LONGITUDINAL STABILITY DERIVATIVES

Derivative		Lifting surface (computer; Rigid and equiv. elas.			Lifting line (computer) Rigid and equiv. elas.			Handbook Rigid only			Handbook + lifting surface Equiv. elas.		
		Sub		Sup	Sub		Sup	Sub		Sup	Sub		Sup
		707	SST	SST	707	SST	SST	707	SST	SST	707	SST	SST
$\alpha$	$C_{D\alpha}$	P	P	P	P	P		P	P				
	$C_{L\alpha}$	F	G	G	G	G		G	P	G	G	P	P
	$C_{m\alpha}$	G	P	F	F	F		P	P	G	P	P	
$\dot{\alpha}$	$C_{D\dot{\alpha}}$							P	P	P			
	$C_{L\dot{\alpha}}$							P	P	P			
	$C_{m\dot{\alpha}}$							P	P	P			
$u$	$C_{Du}$	P	P	P									
	$C_{Lu}$	G	G	G				P	P	P			
	$C_{mu}$	G	G	G									
$q$	$C_{Dq}$	P	P	P									
	$C_{Lq}$	F	F	F				P	P	P	P	P	P
	$C_{mq}$	F	F	F				P	P	P	P	P	P

G (good)—method compares favorably with experiment, with some exceptions.

F (fair)—method compares favorably with experiment, with exception of some

M,  $\Lambda_{LE}$ , etc.

P (poor)—method does not compare favorably with experiment.

[BLANK];—not calculated.

TABLE 5.— LATERAL-DIRECTIONAL STABILITY DERIVATIVES

Derivative		Lifting surface (computer)			Lifting line (computer)			Handbook			Handbook + lifting surface		
		Rigid and equiv. elas.			Rigid and equiv. elas.			Rigid only			equiv. elas.		
		Sub		Super	Sub		Super	Sub		Super	Sub		Super
		707	SST	SST	707	SST	SST	707	SST	SST	707	SST	SST
$\beta$	$C_{Y\beta}$							F	F	F			
	$C_{l\beta}$							P	P	P			
	$C_{n\beta}$							P	F	P			
$\dot{\beta}$	$C_{Y\dot{\beta}}$							P	P	P			
	$C_{l\dot{\beta}}$							P	P	P			
	$C_{n\dot{\beta}}$							P	P	P			
$p$	$C_{Yp}$							P	P		P	P	P
	$C_{lp}$	F	F	F	F	F		P	P		P	P	P
	$C_{np}$							P	P		P	P	P
$r$	$C_{Yr}$							P	P	P			
	$C_{lr}$							P	P	P			
	$C_{nr}$							P	P	P			

G (good)—method compares favorably with experiment, with some exceptions.

F (fair)—method compares favorably with experiment, with exception of some

M,  $\Delta_{LE}$ , etc.

P (poor)—method does not compare favorably with experiment.

[BLANK]—not calculated.

TABLE 6. - APPLICABILITY AND LIMITATIONS OF LIFTING SURFACE THEORY

Derivative		Thin-body formulation, thickness ratio $\delta \ll \bar{c}, b/2$						Non-coplanar, thick-body formulation														
		Wing			Wing-body			Non-coplanar surfaces			Thick surfaces			Thick body			Wing-body Horiz-tail Vert. tail			Wing+tail dihedral		
		S U B	T R A N P	S U P	S U B	T R A N P	S U P	S U B	T R A N P	S U P	S U B	T R A N P	S U B	S U B	T R A N P	S U B	S U B	T R A N P	S U B	S U B	T R A N P	S U P
$\alpha$	$C_{D\alpha}$ $C_{L\alpha}$ $C_{m\alpha}$	•	•	•	•	•	•			•	•		•	•		•	•		•	N	N	N
$\dot{\alpha}$	$C_{D\dot{\alpha}}$ $C_{L\dot{\alpha}}$ $C_{m\dot{\alpha}}$	•		•	•		•	?		?	•		•	•		•				N	N	N
$u$	$C_{Du}$ $C_{Lu}$ $C_{mu}$	•		•	•		•	•		•	•		•	•		•	•		•	N	N	N
$q$	$C_{Dq}$ $C_{Lq}$ $C_{mq}$	•		•	•		•	•		•	•		•	?		?	?		?	N	N	N
$\beta$	$C_{Y\beta}$ $C_{l\beta}$ $C_{n\beta}$	?		?	?		?	?		?	?		?	?		?	?		?	?		?
$\dot{\beta}$	$C_{Y\dot{\beta}}$ $C_{l\dot{\beta}}$ $C_{n\dot{\beta}}$	?		?	?		?															
$p$	$C_{Yp}$ $C_{lp}$ $C_{np}$	•		•	•		•	•		•	•		•									
$r$	$C_{Yr}$ $C_{lr}$ $C_{nr}$	?		?	?		?															

Ability to calculate various flow conditions	
Flow	Capability
Compressible	Yes
Steady	Yes
Quasi-steady	Yes
Unsteady	Yes
Viscous	No
Separated	No

- Capability exists or can be developed from existing theory.
- N Not applicable to first-order approximations of this derivative.
- Blank No capability exists.
- ? Capability may exist, but further development is required.

TABLE 7.- APPLICABILITY AND LIMITATIONS OF LIFTING LINE THEORY

Derivative					Wing-body horiz. tail			Non- coplanar surfaces			Wing-body horiz-tail vert. tail			Wing-tail dihedral			Thick surfaces			Thick body		
		S U B	T R A N S	P	S U B	T R A N S	P	S U B	T R A N S	P	S U B	T R A N S	P	S U B	T R A N S	P	S U B	T R A N S	P	S U B	T R A N S	P
$\alpha$	$C_{D\alpha}$ $C_{L\alpha}$ $C_{m\alpha}$	•			•			•						N	N	N						
$\dot{\alpha}$	$C_{D\dot{\alpha}}$ $C_{L\dot{\alpha}}$ $C_{m\dot{\alpha}}$				•			•						N	N	N						
u	$C_{Du}$ $C_{Lu}$ $C_{mu}$	•			•			•						N	N	N						
q	$C_{Dq}$ $C_{Lq}$ $C_{mq}$	•			•			?						N	N	N						
$\beta$	$C_{Y\beta}$ $C_{l\beta}$ $C_{n\beta}$	•			•			?						?								
$\dot{\beta}$	$C_{Y\dot{\beta}}$ $C_{l\dot{\beta}}$ $C_{n\dot{\beta}}$	?			?			?														
p	$C_{Yp}$ $C_{lp}$ $C_{np}$	?			?			?														
r	$C_{Yr}$ $C_{lr}$ $C_{nr}$	?			?																	

Ability to calculate various flow conditions	
Flow	Capability
Compressible	Yes
Steady	Yes
Quasi-steady	Yes
Unsteady	Yes
Viscous	No
Separated	No

- Capability exists or can be developed from existing theory.
- N Not applicable to first order approximations of this derivative.
- Blank No capability exists.
- ? Capability may exist, but further development is required.

TABLE 8.—APPLICABILITY AND LIMITATIONS OF THE HANDBOOK METHODS

Derivative	Wing			Wing-body			Non-coplanar surfaces			Thick surfaces			Thick body			W-B-Ht vert. tail			Wing-tail dihedral		
	S	T		S	T		S	T		S	T		S	T		S	T		S	T	
	U	A	B	U	A	B	U	A	B	U	A	B	U	A	B	U	A	B	U	A	B
$\alpha$	$C_{D\alpha}$																		N	N	N
	$C_{L\alpha}$	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•			
	$C_{m\alpha}$																				
$\dot{\alpha}$	$C_{D\dot{\alpha}}$									•		•									
	$C_{L\dot{\alpha}}$	•	•	•	•	•	•	•	•		•		•	•	•	•	•	•	N	N	N
	$C_{m\dot{\alpha}}$	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•			
u	$C_{Du}$																		N	N	N
	$C_{Lu}$																				
	$C_{mu}$																				
q	$C_{Dq}$																		N	N	N
	$C_{Lq}$	•	•	•	•	•	•	•	•		•		•	•	•	•	•	•			
	$C_{mq}$	•	•	•	•	•	•	•	•		•		•	•	•	•	•	•			
$\beta$	$C_{Y\beta}$																				
	$C_{l\beta}$	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
	$C_{n\beta}$																				
$\dot{\beta}$	$C_{Y\dot{\beta}}$																				
	$C_{l\dot{\beta}}$									•	•	•	•	•		•	•				
	$C_{n\dot{\beta}}$																				
p	$C_{Yp}$					•									•				•	•	•
	$C_{lp}$	•	•	•						•	•	•			•	•	•	•			
	$C_{np}$					•									•						
r	$C_{Yr}$	•		•															•	•	•
	$C_{lr}$	•		•						•	•	•				•	•	•			
	$C_{nr}$	•	•	•															•	•	•

Ability to calculate various flow conditions	
Flow	Capability
Compressible	Yes
Steady	Yes
Quasi-steady	Yes
Unsteady	Yes
Viscous	No
Separated	No

- Capability exists or can be developed from existing theory.
- N Not applicable to first order approximations of this derivative.
- Blank No capability exists.
- ? Capability may exist, but further development is required.

The preference for lifting surface theory expressed in this report is heavily supported by the results presented in these tables. Three classes of capability are shown. That which exists or can be acquired readily is shown by a dot. That which is within the fundamental limitations of the technique is shown by a question mark. A blank space indicates capability which exceeds fundamental limitations of the technique.

Estimation techniques using lifting surface theory clearly have the greatest basic potential. When this theoretical capability is coupled with empirical corrections to the lifting surface technique, the preference is further justified. Empirical correction methods may be utilized in the aerodynamic influence coefficient methods in a very efficient and rational manner. This feature has been incorporated into existing computer programs based on lifting line theory but is currently in only a rudimentary stage of development in existing programs based on lifting surface theory.

The formulation of stability derivatives in terms of aerodynamic influence coefficients involves the theory of small perturbation inviscid fluid flow. A general formulation of the aerodynamics of stability and control based on a mathematical theory of fluid dynamics appears nowhere in the literature. Many specialized investigations may be found, and examples appear in the works by Miles (ref. 28), Chester (ref. 29), Ward (ref. 30), and Van Dyke (ref. 31). The only general development found in the literature is that by Bryson (ref. 32), which applies to slender rigid bodies. However, Bryson's development is valid for slender bodies only, to the exclusion of wing-body combinations with aspect ratios larger than 2 or 3, since his formulation is based on the constant-density, cross-flow assumption.

It is also important to note that the assumptions applied to the conservation laws of ref. 33 to develop the inviscid fluid dynamic equations do not apply to hypersonic flow. Precise limits of applicability cannot be placed upon these equations, because the onset of hypersonic flow depends upon the shape of the body (ref. 34). Hypersonic flow characteristics are observed at  $M_\infty = 3.0$  on some blunt bodies, but not until  $M_\infty = 10.0$  on some thin bodies. For supersonic jet-type airplanes, which are relatively slender, important hypersonic flow effects do not occur below  $M_\infty = 5.0$ . Therefore, the flow equations used in this report are assumed valid up to that speed for such slender configurations.

A formulation of the stability derivatives in terms of lifting surface aerodynamic influence coefficients is presented in par. 7.2. The basis for that formulation is described in physical terms in par. 7.2 and is developed in detail in app. B. Summarizing, the pertinent equations for the stability derivatives are written in terms of five aerodynamic matrices,  $[A_1]$ ,  $[A_2]$ ,  $[A_3]$ ,  $[A_4]$ , and  $[A_5]$ , which are derived from the equations of fluid dynamics as described briefly in par. 7.2 and in detail in app. B. Their precise form depends, in part, on the particular aerodynamic influence coefficient theory chosen. The equations in terms of the aerodynamic matrices are as follows.

$$\begin{bmatrix} C_{D_u} & \boxed{C_{y_\beta}} & C_{D_\alpha} & \boxed{C_{y_p}} & C_{D_q} & \boxed{C_{y_r}} \\ C_{L_u} & \boxed{C_{l_\beta}} & C_{L_\alpha} & \boxed{C_{l_p}} & C_{L_q} & \boxed{C_{l_r}} \\ C_{m_u} & \boxed{C_{n_\beta}} & C_{m_\alpha} & \boxed{C_{n_p}} & C_{m_q} & \boxed{C_{n_r}} \end{bmatrix} \equiv [\bar{\phi}]^T [A_1] \quad (119)$$

$$\begin{bmatrix} \boxed{C_{y_\beta}} \\ \boxed{C_{l_\beta}} \\ \boxed{C_{n_\beta}} \end{bmatrix} \begin{matrix} C_{D_{\dot{\alpha}}} \\ C_{L_{\dot{\alpha}}} \\ C_{m_{\dot{\alpha}}} \end{matrix} \equiv [\bar{\phi}]^T [A_2] \quad (120)$$

The derivatives contained in the boxes above are lateral-directional; the rest are longitudinal. The aerodynamic influence coefficients cannot be used to compute drag due to viscous forces. However, those drag derivatives which are for the most part due to nonviscous forces can be computed. The values of  $C_{D_u}$ ,  $C_{D_\alpha}$ , and  $C_{D_q}$  can be computed for use in obtaining elastic corrections to rigid values obtained separately from a computer program system.

Equations (119) and (120) contain certain approximations of the aerodynamic forces in order to be consistent with the small perturbation equations of motion. The lateral-directional unsteady derivatives  $C_{y_\beta}$ ,  $C_{l_\beta}$ ,  $C_{n_\beta}$  contain the most severe approximations and assumptions. The longitudinal steady derivatives appearing in equation (119) contain the least severe approximations. Besides those assumptions previously listed in deriving the small perturbation equations of motion, assumptions (D11) and (A8), are



applied in calculating the longitudinal stability derivatives. The evaluation of the lateral-directional stability derivatives depends upon (A12). Unsteady stability derivatives are based on approximation (A9) in lieu of (A8).

Additional stability derivatives are required which correspond to aerodynamic forces generated by the dynamic-elastic motion, i.e., motion in the elastic degrees of freedom. Stability derivatives corresponding to the  $i^{\text{th}}$  elastic degree of freedom are

$$\begin{bmatrix} C_{D_i} \\ C_{y_i} \\ C_{L_i} \\ C_{\ell_i} \\ C_{m_i} \\ C_{n_i} \end{bmatrix} \equiv [\bar{\phi}]^T [A_3] \{\phi_i\} \quad (121)$$

$$\begin{bmatrix} C_{D_i^*} \\ C_{y_i^*} \\ C_{L_i^*} \\ C_{\ell_i^*} \\ C_{m_i^*} \\ C_{n_i^*} \end{bmatrix} \equiv [\bar{\phi}]^T [A_4] \{\phi_i\} \quad (122)$$

$$\begin{bmatrix} C_{D_i} \\ C_{Y_i} \\ C_{L_i} \\ C_{l_i} \\ C_{m_i} \\ C_{n_i} \end{bmatrix} = [\bar{\phi}]^T [A_5] \{\phi_i\} \quad (123)$$

where  $\{\phi_i\}$  is the  $i^{\text{th}}$  elastic free vibration modal shape.

The derivatives appearing in boxes are lateral-directional and are nonzero if  $i$  corresponds to an antisymmetric mode shape and zero if  $i$  corresponds to a symmetrical mode shape. The remaining derivatives are longitudinal and are zero or nonzero conversely with the lateral-directional derivatives.

Stability derivatives for changes in generalized aerodynamic forces in the elastic degrees of freedom due to changes in the motion velocity components are given by

$$\begin{bmatrix} C_{i_u} \\ C_{i_\beta} \\ C_{i_\alpha} \\ C_{i_p} \\ C_{i_q} \\ C_{i_r} \end{bmatrix}^T = \{\phi_j\}^T [A_1] \quad (124)$$

Stability derivatives for changes in generalized aerodynamic forces due to rates of change in incidence and sideslip are:

$$\begin{bmatrix} C_{j\dot{\beta}} \\ C_{j\dot{\alpha}} \end{bmatrix}^T = \{\phi_j\}^T [A_2] \quad (125)$$

Stability derivatives relating changes in generalized aerodynamic forces to changes in the generalized coordinates and velocities in the elastic degrees of freedom are:

$$\{C_{j_i}\} \equiv \{\phi_j\}^T [A_3] \{\phi_i\} \quad (126)$$

$$\{C_{j_i}\} \equiv \{\phi_j\}^T [A_4] \{\phi_i\} \quad (127)$$

Stability derivatives relating changes in generalized aerodynamic forces to changes in the generalized accelerations in the elastic degrees of freedom are:

$$\{C_{j_i}\}^T \equiv \{\phi_j\}^T [A_5] \{\phi_i\} \quad (128)$$

The derivatives in equation (128) may be computed on the basis of aerodynamic influence coefficients for reduced frequencies that are small and for symmetrical mode shapes.

In summary, equations (119) through (128) contain all the stability derivatives for a symmetric, elastic airplane. If the appropriate aerodynamic influence coefficients are available, then all of the stability derivatives may be computed.

The stability derivatives computed on the basis of aerodynamic influence coefficients represent a valid first-order approximation for the pressure coefficient of an inviscid, isentropic fluid flow (ref. 30). In addition, it is assumed that the unsteady effects are due to motion of sufficiently slow variation that the reduced frequency is of zero order of magnitude (ref. 28).

## 7.2 Theoretical Methods

Stability derivatives relate changes in the aerodynamic forces and moments on an airplane to changes in airplane attitude and motion. This section describes a basis for computing the stability derivatives from the mathematical theory of fluid flow. The aerodynamic influence method is used in the development. Results sought are the aerodynamic matrices  $[A_1]$ ,  $[A_2]$ ,  $[A_3]$ ,  $[A_4]$ , and  $[A_5]$ , which appear in the matrix formulations of the equations of motion appearing in Sec. 5. The detailed developments are contained in app. B. This section merely summarizes and discusses the physical aspects of the problem, but proceeds from rather fundamental concepts.

An airplane in flight may be looked upon as a solid body in motion in a fluid. Its motion disturbs a portion of the atmosphere. However, its shape and motion are usually such that the perturbation of the disturbed fluid is very slight. In fact, the magnitude of perturbation velocity of the fluid, when divided by the magnitude of the airplane's velocity, is of order of magnitude zero except in the vicinity of stagnation points. Also, the viscosity of the atmosphere is so small that viscous forces are negligible in comparison with dynamic forces in the fluid, except in a thin layer of flow at the airplane's surface and in its wake. These conditions allow small perturbation inviscid flow theory to provide an accurate theoretical representation of the flow about an airplane.

As shown in app. B, as well as in most textbooks on aerodynamics, in the theory of small perturbation inviscid flow the principles of conservation of mass and momentum and the equation of state are reduced to a single linear partial differential equation. This flow equation contains a single dependent variable, perturbation velocity potential  $\phi$ . The flow equation is written in terms of a fluid axis system whose coordinates, along with time, are the independent variables in the flow equation.

The velocity of the fluid is given by  $\vec{V} = U(\vec{i} + \vec{\nabla} \phi)$ . This is the velocity of the fluid particles measured relative to the fluid axis system and  $U$  is the component, along the x-axis of the fluid axis system, of the velocity of the airplane relative to fixed space.

The perturbation velocity potential must be such that it satisfies the flow equation and such that the velocity field  $\vec{V}(x,y,z,t)$  does not produce flow through the surface of the airplane. This last requirement is called a boundary condition. An additional requirement of the perturbation velocity potential follows from the vortex laws of Helmholtz. The wake behind the airplane must be such that they are satisfied. However, the pressure in the fluid calculated from Bernoulli's equation must be continuous across the wake. Discontinuities in the pressure field in the fluid can only exist across solid bodies. This is all that will be said about the flow problem here; for a more detailed description, the reader may refer to app. B. The special form appropriate to the question of aerodynamic forces in stability and control considerations will now be introduced.

The boundary condition at the surface (see app. B) may be written as

$$\vec{n} \cdot (\vec{V}_s - \vec{u}) = 0 \quad (129)$$

where  $\vec{n}$  is a vector normal to the surface of the airplane. The velocities  $\vec{V}_s$  and  $\vec{u}$  are the velocities of fluid particles and elastic particles of the airplane at the surface. They are measured relative to the stability axis system that rotates and translates with the airplane ( $X_s, Y_s, Z_s$ ).

The stability axis system moves relative to the fluid axis system with the translational velocity  $\vec{V}_R = V\vec{j}_s + W\vec{k}_s$  and rotational velocity  $\vec{\omega} = p\vec{i}_s + q\vec{j}_s + r\vec{k}_s$ . Thus, the two fluid velocities  $\vec{V}$  and  $\vec{V}_s$  at the surface of the airplane are related as

$$\vec{V} = \vec{V}_s + \vec{V}_R + \vec{\omega} \times \vec{r}_s \quad (130)$$

where  $\vec{r}_s$  is the position vector of a point on the surface relative to the center of gravity, the origin of the stability axis system.

Using equation (130), the boundary condition at the airplane surface becomes

$$\vec{n} \cdot [U\vec{i} + U\vec{\nabla}\phi - (Qz_s - Ry_s)\vec{i}_s - (V + Rx_s - pz_s)\vec{j}_s - (\omega + py_s - Qx_s)\vec{k}_s - \vec{a}] = 0 \quad (131)$$

where it is important to note that some quantities are described in the fluid axis system while others are described in the stability axis system. Also, each of the velocity components appearing in equation (131) is treated as an independent variable in the equations of motion, Sec. 5. Further, the perturbation velocity  $\vec{\nabla}\phi$  must be of order of magnitude zero. Thus, each individual term involving a velocity component when divided by  $U$  must be of order of magnitude zero for a valid approximation based on small perturbation, inviscid flow theory. The approximation in the small perturbation equations of motion becomes linked to those in the flow theory through this boundary condition.

In the reference motion of the airplane, which is steady, the boundary condition reduces to

$$\begin{aligned} \vec{n}_1 \cdot \vec{\nabla}\phi_1 = \frac{\vec{n}_1}{U_1} \cdot [ & U_1\vec{i}_s + (Q_1z_s - R_1y_s)\vec{i}_s + (V_1 + R_1x_s)\vec{j}_s \\ & - Q_1x_s\vec{k}_s + \vec{a}_1] \end{aligned} \quad (132)$$

In the disturbed motion, the airplane may be so oriented relative to fixed space that the Euler angles are perturbed as  $\psi, \Theta, \phi$  from those of the reference motion. These variables are treated as independent, so that their perturbations are independent of the velocity perturbations. The flow problem must be reoriented with the airplane. The only change in the boundary condition resulting from the orientation perturbation arises from the elastic deformation perturbation due to the perturbation of the gravity forces. Thus, there is a change in the direction of the normal vector  $\vec{n}$  relative to the stability axis system, but there is no other change.

Neglecting products of perturbation quantities and using the reference motion boundary condition, the perturbed boundary condition is found to be

$$\begin{aligned}\bar{\mathbf{n}}_1 \cdot \bar{\nabla} \phi_p = & \left[ \frac{u_s}{U_1} + \beta \bar{\mathbf{j}}_s + \alpha \bar{\mathbf{k}}_s + \left( \frac{qz_s}{U_1} - \frac{ry_s}{U_1} \right) \bar{\mathbf{i}}_s \right. \\ & + \left. \left( \frac{rx_s}{U_1} - \frac{pz_s}{U_1} \right) \bar{\mathbf{i}}_s + \left( \frac{py_s}{U_1} - \frac{qx_s}{U_1} \right) \bar{\mathbf{k}}_s + \frac{\bar{d}_p}{U_1} \right] \cdot \mathbf{n}_1 \quad (133) \\ & + \delta \bar{\mathbf{n}}_1 \cdot \bar{\mathbf{i}}_s\end{aligned}$$

The velocity potential  $\phi_p$  is the perturbation to the perturbation velocity potential in the reference motion such that  $\phi = \phi_1 + \phi_p$ . The vector  $\delta \bar{\mathbf{n}}_1$  is the change in the vector normal to the surface caused by the change in elastic shape of the airplane from reference to disturbed motion. To a first-order approximation, it is given by the elastic rotation at the surface  $\bar{\Theta}_E$  as

$$\delta \bar{\mathbf{n}}_1 = \bar{\Theta}_E \times \bar{\mathbf{n}}_1 \quad (134)$$

and  $\bar{\Theta}_E$  is due to the perturbed gravitational, inertial, and aerodynamic forces. It must be found from the initial equilibrium equations of Sec. 5, as must the perturbed elastic displacement rate,  $\bar{u}_p$ .

The velocity component  $U \bar{\mathbf{n}} \cdot \bar{\nabla} \phi$  is the perturbation to the flow from its freestream direction that is required to satisfy the slope boundary condition at the airplane's surface. In general, the perturbed flow may be unsteady, but the angle between the direction of the streamlines of the undisturbed flow and the streak lines of the disturbed flow is defined by  $\tan \psi = \bar{\mathbf{n}} \cdot \bar{\nabla} \phi$  at the airplane's surface. This is the local incidence angle at any instant of time and because it is small

$$\psi \approx \bar{\mathbf{n}} \cdot \bar{\nabla} \phi \quad (135)$$

By definition of the reference motion, the incidence angle does not change with time in the reference motion, so that

$$\psi_1 \approx \bar{\mathbf{n}}_1 \cdot \bar{\nabla} \phi_1 \quad (136)$$

is the angle between unperturbed and perturbed streamlines. The perturbation, local incidence angle

$$\psi_p \approx \vec{n}_1 \cdot \vec{\nabla} \phi_p \quad (137)$$

is in general a function of time.

The local flow incidence angles must be related to aerodynamic pressure at the airplane's surface before the above results may be used to generate stability derivatives. Surface pressures are readily computed from the velocity potential using an appropriate small perturbation form of Bernoulli's equation as shown in app. B. Thus, the problem centers on finding perturbation velocity potentials which satisfy the flow equation and the boundary conditions.

The most widely used means for finding perturbation velocity uses the linearity of the flow equation in its small perturbation form. Elementary solutions to the flow equation are obtained. These are perturbations to uniform flows produced by sources and doublets. Their distribution throughout the flow and their strengths are so adjusted that the uniform flow is perturbed into conformance with the airplane's shape and motion. The sources and doublets are located in the volume that is interior to the airplane and doublets are distributed in the wake. In lieu of the solid body, an interior flow is formed; the boundary between this interior flow and the exterior flow is made to conform to the airplane surface. Further, the circulation required by the Kutta condition is imposed and the wake is made to satisfy the vortex laws.

The linearized flow equation is essential to this type of flow representation. The linear combination of the effects of the sources and doublets is only possible if the flow equation is a linear partial differential equation.

In practice, the flow boundary condition cannot, in general, be satisfied at every point of the airplane's surface. A finite number of surface points are chosen as control points, and the boundary condition is satisfied at those points to within some chosen numerical accuracy. At points in between, the boundary condition may or may not be satisfied. However, when the number of control points is large and the surface is a smooth aerodynamic surface, the approximation may be of high quality.

Additional approximations may be introduced. These are associated with a linearization of the boundary conditions in a manner consistent with the linearization of the flow equation. As shown in app. B, for thin bodies such as wings and tail surfaces the thickness may be considered separately from the incidence. For slender bodies such as fuselages the axial component of flow may be considered separately from the cross-flow component.

The thickness of thin bodies may be represented by a distribution of sources at a middle surface, and incidence is represented by distributed doublets. For the axial flow over slender bodies, line distributions of sources are placed on the fuselage axis. The effect of cross flow is met by line doublets along the fuselage axis.

There are several schemes used in distributing the sources and doublets. They may have specified distributions over small regions called panels. In this case their strengths may be constant or have some specified variation over the surface of the panel. The distributed doublets on the panels are vortex sheets. In the method by Woodward, designated as computer program TA-67A in this report (ref. 35), the Kutta condition is satisfied along with the vortex laws on each panel (see app. B). This method does not require separate treatment of the wake.

An alternate scheme for treating the vorticity distribution on thin bodies utilizes a vorticity distribution represented by a series of loading functions that vary over the entire middle surface. The coefficients of the series are adjusted so as to satisfy the flow boundary condition at a finite number of control points. This is a collocation method; numerous other methods have been used. Reference to the books by Ashley and Landahl (ref. 27) and Ward (ref. 30) will provide an introduction to many of the alternate approaches.

The approach used by Woodward (ref. 35) works well with the method that includes the empirical corrections required by stability and control applications. It relates a localized cause to a localized effect so that localized empirical corrections may be introduced. Aerodynamic influence coefficients are generated and admit of the following aerodynamic equation

$$\{F_A\} = \bar{q} ([A] \{\alpha\} + [A_T] \{\alpha_T\}) \quad (138)$$

where the elements  $F_{A_i}$  are the aerodynamic panel pressure forces. The elements  $\alpha_i$  are the incidence angles of the panels and  $\alpha_{T_i}$  are the panel incidences due to thickness. The matrices  $[A]$  and  $[A_T]$  are matrices of the corresponding aerodynamic influence coefficients.

Up to this point the discussion of the aerodynamic problem has ignored the effects of unsteady flow; the result represented by equation (138) is valid only for steady or quasi-steady flow. As shown in app. B, however, an approach suggested in Miles' monograph (ref. 28) may be used to include the effects of unsteady flow in a manner appropriate to the aerodynamics of stability and control. This approach leads to a direct extension of equation (138) when the time dependence of the flow incidence angles is slowly varying. As shown in app. B, the time variation must be such that its reduced frequency is in the region of reduced frequencies of order of magnitude zero. This is a good approximation for consideration of the stability characteristics of large, flexible airplanes even though it is not satisfactory for flutter problems.



The consequence of mechanizing this approach following the method by Woodward will lead to an aerodynamic equation of the form

$$\{F_A\} = \bar{q} ([A] \{\psi\} + [\delta A] \{\dot{\psi}\} + [A_T] \{\psi_T\}) \quad (139)$$

The elements of  $[\delta A]$  will relate the panel pressure forces to the rates of panel incidence change.

By evaluating the boundary condition given by equations (132) through (136) at the aerodynamic control points and substituting it into equation (139), a complete formulation of the aerodynamic problem of stability and control is obtained. In the reference motion case, time dependence drops from consideration, and the pressure force due to thickness enters the problem only through the elastic deformations. However, an additional consideration is necessary in the case of disturbed motion. This involves the coordinate axis systems used to describe the problem.

The aerodynamic equation, equation (138) or (139), is derived in terms of the fluid axis system. It may be recalled that the freestream velocity,  $U$ , and the Mach number,  $M$ , are perturbed. The fluid axis system in the reference motion differs from that in the disturbed motion. The aerodynamic equation is perturbed, and this may be expressed by writing equation (139) as

$$\begin{aligned} \{F_{A_1}\} + \{f_A\} = (\bar{q}_1 + \bar{q}_p) & \left[ \left( [A_1] + \left[ \frac{\partial A}{\partial M} \right] M_1 \frac{u}{U_1} \right) (\{\psi_1\} + \{\psi_p\}) \right. \\ & \left. + [\delta A] \{\dot{\psi}\} + \left( [A_{T_1}] + \left[ \frac{\partial A_T}{\partial M} \right] M_1 \frac{u}{U_1} \right) \{\psi_T\} \right] \quad (140) \end{aligned}$$

By discarding products of perturbation quantities and using the form of equation (139) for the reference motion, it follows that

$$\begin{aligned} \{f_A\} = \bar{q}_1 & \left\{ \left[ 2[A_1] + M_1 \left[ \frac{\partial A}{\partial M} \right] \right] \{\psi_1\} + \left[ 2[A_{T_1}] + M_1 \left[ \frac{\partial A_T}{\partial M} \right] \right] \{\psi_T\} \right\} \frac{u}{U_1} \\ & + \bar{q}_1 [A_1] \{\psi_p\} + \bar{q}_1 [\delta A] \{\dot{\psi}_p\} \quad (141) \end{aligned}$$

This expresses the perturbation to the aerodynamic forces in terms of the perturbation motion variables. The first-order approximation,  $\bar{q}_p \approx 2q_1 u/U_1$ , has been used. Also, it may be noted that the first term contains products of perturbation quantities since the reference motion incidence angles must be of perturbation order. The justification of this is somewhat loose. It falls on the requirements of stability and control. The  $u$  perturbation variable must be treated independently of the other variables. The first term, which contains  $u$  solely, must be considered separately in the order of magnitude consideration.

This last consideration points up a problem that requires a great deal of further investigation. Equation (141) has been written nearly a priori, but the basis of the equation is far from self evident. A very careful and fundamental order-of-magnitude analysis is required in order to obtain a consistent, valid formulation of the influence coefficients in the equation. This is illustrated by the development of the results, which appear in app. B. A derivation of equation (141) is carried out for a thin body undergoing longitudinal perturbations. The basis for a source extension of that analysis to thin wing, slender body combinations undergoing longitudinal disturbances also appears to exist (refs. 29 and 35). However, the lateral-directional problem for wing-body combinations has not been developed. An intuitive approach to developing a computer program for solving this lateral-directional problem might be instituted. However, without an order-of-magnitude analysis there would be no way to assess the limitations of such a program. A failure of the program to accurately predict lateral-directional derivatives might defy further intuitive judgment and a large engineering effort might be needlessly lost in numerical experimentation.

In the preceding equations, (133) through (141), elastic deformations must be expressed in terms of the free vibration mode shapes. The perturbation to the elastic displacement vector is given by

$$\{\dot{d}_p\} = [\phi] \{u_p\} \quad (142)$$

where the generalized perturbation displacements  $u_{pi}$  are functions of time alone. The displacement rate  $\dot{d}_p$  in equation (133), therefore, can be written in matrix form as

$$\{\dot{d}_p\} = [\phi] \{\dot{u}_p\} \quad (143)$$

The triple scalar product  $(\Theta_{E_p} \times \bar{n}_1) \cdot \bar{i}_s$  may be written as

$$\left( \bar{i}_s \times \bar{\Theta}_{E_p} \right) \cdot \bar{n}_1 = \left( -\psi_{E_p} \bar{j}_s + \Theta_{E_p} \bar{k}_s \right) \cdot \bar{n}_1 \quad (144)$$

where  $\psi_{E_p}$  and  $\Theta_{E_p}$  are the components of elastic rotation about the  $z_s$  and  $y_s$  axes, respectively. Two additional modal matrices are introduced such that

$$\left\{ \psi_{E_p} \right\} = [\phi_\psi] \{u_p\} \quad \text{and} \quad \left\{ \Theta_{E_p} \right\} = [\phi_\Theta] \{u_p\} \quad (145)$$

These represent the elastic slope deflections required for equation (144), but it must be noted that the free vibration mode shapes do not describe the elastic deformation completely when the mode shapes are generated in the manner described in app. A. The elastic deflections arising from perturbation changes to the orientation of the gravity force and rigid-body inertial forces must be introduced separately. This is described in the discussion leading to equation (6.86b) in app. A as well as in the discussion given in par. 6.3.4 of app. A.

Denote the elastic rotations arising from inertial relief and gravity perturbation forces as

$$\left\{ \psi'_{E_p} \right\} \quad \text{and} \quad \left\{ \Theta'_{E_p} \right\}$$

The elastic rotations may be found from specialized forms of the flexibility matrix denoted by  $[\bar{C}_\psi]$  and  $[\bar{C}_\Theta]$ . Then, in accordance with equation (6.165) of app. A

$$\begin{aligned} \left\{ \psi'_{E_p} \right\} &= -[\bar{C}_\psi] [\bar{m}] [\bar{\phi}] \left( \frac{\partial}{\partial t} \{V_p\} + [M_1] \{V_p\} + [M_2] \{r'_p\} \right) \\ \text{and} \quad \left\{ \Theta'_{E_p} \right\} &= -[\bar{C}_\Theta] [\bar{m}] [\bar{\phi}] \left( \frac{\partial}{\partial t} \{V_p\} + [M_1] \{V_p\} + [M_2] \{r'_p\} \right) \end{aligned} \quad (146)$$

Further, letting the matrices of the components of the normal vectors  $\bar{n}_{1_i}$  be defined as

$$[\bar{n}_1] \cdot \bar{i}_s \equiv [n_x], \quad [\bar{n}_1] \cdot \bar{j}_s \equiv [n_y], \quad [\bar{n}_1] \cdot \bar{k}_s \equiv [n_z]$$

the expression for the perturbed flow incidences, equation (143), may be written in matrix form as

$$\{\psi_p\} = [n_x] \left( \{1\} \frac{u}{U_1} + \{z_s\} \frac{q}{U_1} - \{y_s\} \frac{r}{U_1} + \frac{1}{U_1} [\phi_x] \{u_p\} \right) \quad (147)$$

$$\begin{aligned}
& + [\mathbf{n}_y] \left( \{1\} \beta + \{x_s\} \frac{r}{U_1} - \{z_s\} \frac{p}{U_1} - [\phi_x] \{u_p\} - \left\{ \psi'_{E_p} \right\} + \frac{1}{U_1} [\phi_y] \{\dot{u}_p\} \right) \\
& + [\mathbf{n}_z] \left( \{1\} \alpha + \{y_s\} \frac{p}{U_1} - \{x_s\} \frac{q}{U_1} + [\phi_\Theta] \{u_p\} + \left\{ \Theta'_{E_p} \right\} + \frac{1}{U_1} [\phi_z] \{\dot{u}_p\} \right)
\end{aligned} \tag{147}$$

By introducing appropriate matrix definitions, equation (147) may be written as

$$\{\psi_p\} = [G_1] \{V_p\} + [G_2] \{u_p\} + [G_3] \{\dot{u}_p\} - [\mathbf{n}_y] \left\{ \psi'_{E_p} \right\} + [\mathbf{n}_z] \left\{ \Theta'_{E_p} \right\} \tag{148}$$

where

$$\{V_p\}^T = [u, v, w, p, q, r]$$

as in Sec. 5.

The perturbation aerodynamic equation, equation (141), may be written in an abbreviated form by letting

$$[\bar{A}_1] \equiv \left[ 2 [A_1] + M_1 \left[ \frac{\partial A}{\partial M} \right] \right] \{\psi_1\} + \left[ 2 [A_{T_1}] + M_1 \left[ \frac{\partial A_T}{\partial M} \right] \right] \{\psi_T\} \tag{149}$$

so that on introducing equation (148)

$$\begin{aligned}
\{f_A\} = & \bar{q}_1 \{\bar{A}_1\} \frac{u}{U_1} + \bar{q}_1 [A_1] [G_1] \{V_p\} + \bar{q}_1 [\delta A] \{\dot{V}_p\} \\
& + \bar{q}_1 [A_1] [G_2] \{u_p\} + \bar{q}_1 \left[ [A_1] [G_2] + [\delta A] [G_2] \right] \{\dot{u}_p\} \\
& + \bar{q}_1 [\delta A] [G_3] \{\ddot{u}_p\} + \bar{q}_1 [A_1] \left( -[\mathbf{n}_y] \left\{ \psi'_{E_p} \right\} + [\mathbf{n}_z] \left\{ \Theta'_{E_p} \right\} \right)
\end{aligned} \tag{150}$$

where the effects of time rates of change of  $\{\psi' E_p\}$  and  $\{\theta' E_p\}$  have been considered negligible. The aerodynamic matrices introduced in the equations may now be recognized as

:

$$[A_1] \equiv \bar{q}_1 \left[ \{\bar{A}_1\} \frac{1}{U_1}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\} \right] + \bar{q}_1 [A_1] [G_1] \quad (151a)$$

$$[A_2] \equiv \bar{q}_1 \left[ [\delta A] + [A_1] \left( -[n_y] [\dot{\bar{C}}_\psi] + [n_z] [\dot{\bar{C}}_\theta] \right) [m] [\bar{\phi}]^T \right] \quad (151b)$$

$$[A_3] \equiv \bar{q}_1 \left[ [A_1] [G_2] + [A_1] \left( -[n_y] [\dot{\bar{C}}_\psi] + [n_z] [\dot{\bar{C}}_\theta] \right) [m] [\bar{\phi}]^T [M_1] \right] \quad (151c)$$

$$[A_4] \equiv \bar{q}_1 \left[ [A_1] [G_3] + [\delta A] [G_2] \right] \quad (151d)$$

$$[A_5] \equiv \bar{q}_1 [\delta A] [G_3] \quad (151e)$$

Hence, with these definitions equation (150) becomes identical to equation (126) of Sec. 6, i.e.,

$$\begin{aligned} \{f_A\} = & [A_1] \{V_p\} + [A_2] \{\dot{V}_p\} + [A_3] [\phi] \{u_p\} + [A_4] [\phi] \{\dot{u}_p\} \\ & + [A_5] [\phi] \{\ddot{u}_p\} \end{aligned} \quad (152)$$

### 7.3 Semi-Empirical Methods

The semi-empirical or "handbook" methods have evolved from numerous wind tunnel tests and theoretical analyses of various aircraft configurations to evaluate the effects of geometry on the stability derivatives. These techniques, as do both the purely experimental and purely theoretical techniques, seek to solve the exact flow equations of ref. 33. In addition, the handbook techniques seek to improve the theoretical solutions based on inviscid fluid dynamic results by including experimentally measured forces and moments due to leading edge suction, boundary layer, and other nonlinear effects not included in the inviscid approximation of the aerodynamics. The inviscid flow theory is then modified with the experimental results and plotted as functions of aerodynamic and geometric parameters

(ref. 6). An engineer interested in calculating stability derivatives for a particular configuration must seek the appropriate tables and select the separate effects due to wing, body, tail, canard, etc. Appropriate interference factors between wing-body, wing-tail, etc., are also presented in the tables to account for the nonlinearity of the flow field. Appendix B indicates which of the stability derivatives can be calculated using the USAF Stability and Control Handbook.

The procedure used to calculate the stability derivatives by the handbook technique is best illustrated by an example. Consider the calculation of  $C_{L\alpha}$ :

$$C_{L\alpha} = \underbrace{\left(C_{L\alpha}\right)'_e \frac{S'_e}{S'} \left[ K'_N + K'_{W(B)} + K'_{B(W)} \right]'}_{\text{WING-BODY TERM}} + \underbrace{\left(C_{L\alpha}\right)''_e \left[ K''_{W(B)} + K''_{B(W)} \right]''}_{\text{HORIZONTAL TAIL TERM}} \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \frac{\bar{q}}{q_\infty} \frac{S'}{S''} \frac{S''_e}{S''} \quad (153)$$

where  $\left(C_{L\alpha}\right)'_e$  is the lift curve slope for the wing of appropriate aspect ratio, taper ratio, section profile (camber, thickness), twist distribution, etc.

$K'_N$  is the ratio of aircraft nose lift to aircraft wing lift.

$K'_{W(B)}$  is the effect of lift carryover on the wing due to the body.

$K'_{B(W)}$  is the effect of lift carryover on the body due to the wing.

$\left(C_{L\alpha}\right)''_e$  is the lift curve slope of a tail with the appropriate geometric parameters.

$K''_{W(B)}$  is the effect of lift carryover on the tail due to the body.

$K''_{B(W)}$  is the effect of lift carryover on the body due to the tail.

$\frac{\partial \epsilon}{\partial \alpha}$  is the change in downwash on the tail due to the change in angle of attack of the airplane.

$\frac{q''}{q_\infty}$  is the ratio of dynamic pressure at the tail to freestream dynamic pressure.

$\frac{S'}{S_e''}$  is the ratio of wing area to the "effective area" of the tail.

$\frac{S_e''}{S''}$  is the ratio of effective tail area to the actual tail area.

Equation (153) represents summed knowledge based on the personal experience of the authors of the handbook. It should be emphasized that the approximation of  $C_{L\alpha}$  in the equation is not unique because of the "experience factor" involved. Therefore, there is no guarantee that the approximation is correct for all configurations. The handbook technique is the only semi-empirical technique available at this time that can be used to estimate nonlinear effects in the viscous fluid dynamic equations. Accuracy by this technique is only a function of the ingenuity, insight, and experience of the engineer performing the analysis.

One other disadvantage of the handbook technique is that it can only be used to evaluate the rigid and equivalent elastic mathematical models of an elastic aircraft. It cannot be used on the completely elastic and residual elastic models because there is no known way to develop consistent structural derivatives dependent only upon external geometry parameters.

Handbook data for a study SST configuration and the 707-320B are presented in app. B.

## 7.4 Experimental Methods

The two principal experimental methods used to extract the stability derivatives are: (1) wind tunnel tests of a scaled model of the proposed aircraft and (2) flight tests of a similar aircraft or the actual aircraft. Both of the experimental methods are potentially more accurate than either the theoretical or semi-empirical methods, but neither offers a useful and convenient way of optimizing a configuration for good stability and control characteristics.

The main disadvantage of both experimental methods is the high cost of each test point and the difficulty of correcting the collected data for errors due to experimental procedure. In addition, the only derivatives generated by a flight test technique are for completely elastic airplanes, and the extraction of rigid airplane values for comparison with the results of other methods is difficult, if not impossible.

7.4.1 Wind tunnel tests.-- Wind tunnel tests are a reliable way of measuring rigid aircraft static stability derivatives. In addition, several experimental techniques to measure the  $\dot{\alpha}$ ,  $q$ , and  $p$  dynamic stability derivatives are currently under development.

Recently, a new technique of generating the static stability derivatives for an equivalent elastic aircraft was developed and is outlined in app. B. A semi-elastic model is built with structurally and aerodynamically scaled wings and tail. Force and moment data, if properly adjusted for scale effects and tunnel wall effects, can then be reduced to give the

equivalent elastic stability derivatives. These derivatives are then used in the dynamic equations describing the airplane's motion through space. Inertial effects must be included in the analysis to account for load factor changes. :

In some cases the equivalent elastic stability derivatives can be calculated from rigid wind tunnel data. In this method the lifting surface computer program is used to analyze the rigid and elastic aircraft. First, rigid lift curve slopes are calculated; then, using the appropriate structural influence coefficients, the elastic lift curve slopes are calculated. The ratio of  $C_{L\alpha E}/C_{L\alpha R} = L_E/L_R$  is formed and applied to all subsequent rigid wind tunnel data to calculate the effect of elasticity. For some configurations, such as the SST of this study, the effect of elasticity on rigid wind tunnel values can be estimated by an elastic increment (see par. 7.5).

Several difficulties arise in the use of wind tunnel data. Because the model parameters used to evaluate stability derivatives have not yet been outlined, the engineer must determine the correct trip-strip size and location for each stability derivative to duplicate the proper nonlinear viscous effects. The effects of Reynolds number and "rigid model elasticity" are usually unknown. Also, some wind tunnel tests may be plagued with high tunnel turbulence or other forms of wind-tunnel-induced effects in the desired force and moment data.

Data for a study SST configuration and the 707-320B are presented in app. B.

**7.4.2 Flight test techniques.**— The evaluation of stability derivatives from flight test data should, by definition, give the most accurate elastic airplane values. The engineer can use such data in much the same way that he uses the handbook technique, provided individual component contributions to each stability derivative are known or can be extracted from the data measured in flight testing.

The primary problem associated with stability derivative evaluation from flight test data is the difficulty of performing maneuvers holding all motion variables, except one, equal to zero. Obviously, even if it were possible to fly only at an angle-of-attack variation, the aircraft would elastically deform at some frequency determined by the structural properties, and the force and moment data measured at the center of gravity would contain both  $\alpha$  and structural motion ( $d_p$ ) contributions.

In addition, there are a limited number of maneuvers available from which to extract the stability derivatives. The usual methods for the longitudinal stability derivatives are control column steps and pulses, windup turns, and thrust steps and pulses. Lateral derivatives may be extracted from maneuvers arising from steady sideslip and from wheel and rudder-pedal steps and pulses. The resulting forces and moments measured at the



aircraft center of gravity by accelerometers reflect not only aerodynamic effects but also the effects of the pilot's transfer function (lags, response time, insensitivity to small but measurable errors of trim, etc.) and the control system elasticity (cable stretch, internal friction, and nonlinear effects due to the wear of hydraulic systems and control surface linkages). Thus, the stability derivatives extracted from such flight test data also reflect these errors.

Data for the 707-320B are presented in app. B.

## 7.5 Comparison of Methods

Figures 6 through 17 present comparisons of the theoretic, semi-empirical, and experimental evaluations of  $C_{L_\alpha}$ ,  $C_{m_\alpha}$ ,  $C_{m_q}$ ,  $C_{m_u}$ , and  $C_{l_p}$  stability derivatives. The experimentally determined values (wind tunnel and flight test) are to be taken as the basis for comparing accuracy. A comparison of the estimates of all stability derivatives listed in table 3 is contained in app. B. Discussion of the approximations used in the estimates also appears in app. B.

The stability derivative evaluations presented here were chosen because a complete comparison of theoretical and semi-empirical methods could be made against both wind tunnel and flight test data. As shown by tables 6, 7, and 8, estimates are not possible for both techniques for all stability derivatives of interest. Further, testing techniques do not exist or have not been used to evaluate all of these stability derivatives for the 707-320B and the SST configuration. Thus, a basis for comparison does not exist for all of the desired calculations.

Wind tunnel evaluation of  $C_{L_\alpha}$  and  $C_{m_\alpha}$  from both rigid and elastic wind tunnel models is shown in figs. 16 and 17. These data are shown in comparison with estimates obtained from lifting surface theory. The results are functions of both dynamic pressure and Mach number, as noted on the figures. Consequently, the interconnecting lines between values for the elastic model are for visual purposes only; interpolation for intermediate values is not possible.

Primary importance has been placed on the calculation of the static and quasi-steady longitudinal derivatives, since for these derivatives all three techniques are applicable. Four of the six primary longitudinal derivatives are presented in figs. 6 through 13 to compare the techniques. For both the 707 and SST, the lifting surface values of  $C_{L_\alpha R}$  (figs. 6 and 7) are closer than the handbook values to the wind tunnel values. The value of  $C_{m_\alpha R}$  for the 707 (fig. 8) calculated by lifting surface theory is closer to the wind tunnel value than are values calculated by either lifting line or handbook techniques. In the case of the SST,  $C_{m_\alpha R}$  is also calculated more accurately by the lifting surface technique than by the handbook

technique. No experimental data were available to evaluate the accuracy of  $C_{m\dot{q}}R$  (figs. 10 and 11) for the 707 and SST. The value of  $C_{m\dot{u}}R$  (figs. 12 and 13) calculated by lifting surface theory compares very well with wind tunnel data for both the 707 and SST; the handbook technique will not handle this derivative.

The only significant lateral-directional derivative listed in table 3 that can be calculated by both lifting surface and handbook techniques is  $C_{l_p}$ . Figures 14 and 15 show the comparison for  $C_{l_p}R$ ; unfortunately no rigid experimental values were available.

The comparison of the techniques used to calculate elastic stability derivatives is hampered by an inconsistent set of data. Of the five primary derivatives shown in figs. 6 through 15,  $C_{l_{pE}}$  for the 707 (fig. 14) provides the best test of accuracy. Both the handbook and lifting surface techniques give the same value of  $C_{l_{pE}}$  at 3050 meters altitude, but at 10 675 meters the lifting surface technique is more accurate. For this reason, and because lifting surface theory is more accurate for rigid derivatives, it is concluded that the remaining equivalent elastic derivatives will also be calculated more accurately by the lifting surface technique than by the handbook technique.

As additional justification, figs. 16 and 17 show that the incremental error between wind tunnel and lifting surface values of  $C_{L\alpha}$  and  $C_{m\alpha}$  does not increase appreciably between the rigid and the equivalent elastic cases. In fact, figs. 16 and 17 indicate that an empirical correction to the rigid wind tunnel data may result in a more accurate elastic value of the stability derivative. For the study SST at 72° sweep, these corrections are of the form

$$C_{L\alpha_{EqEl}} = \left( C_{L\alpha} \Big|_{\text{Computed}} \right)_{\text{Elastic}} - \left( C_{L\alpha} \Big|_{\text{Computed}} - C_{L\alpha} \Big|_{\text{WT}} \right)_{\text{Rigid}}$$

$$C_{m\alpha_{EqEl}} = \left( C_{m\alpha} \Big|_{\text{Computed}} \right)_{\text{Elastic}} - \left( C_{m\alpha} \Big|_{\text{Computed}} - C_{m\alpha} \Big|_{\text{WT}} \right)_{\text{Rigid}}$$

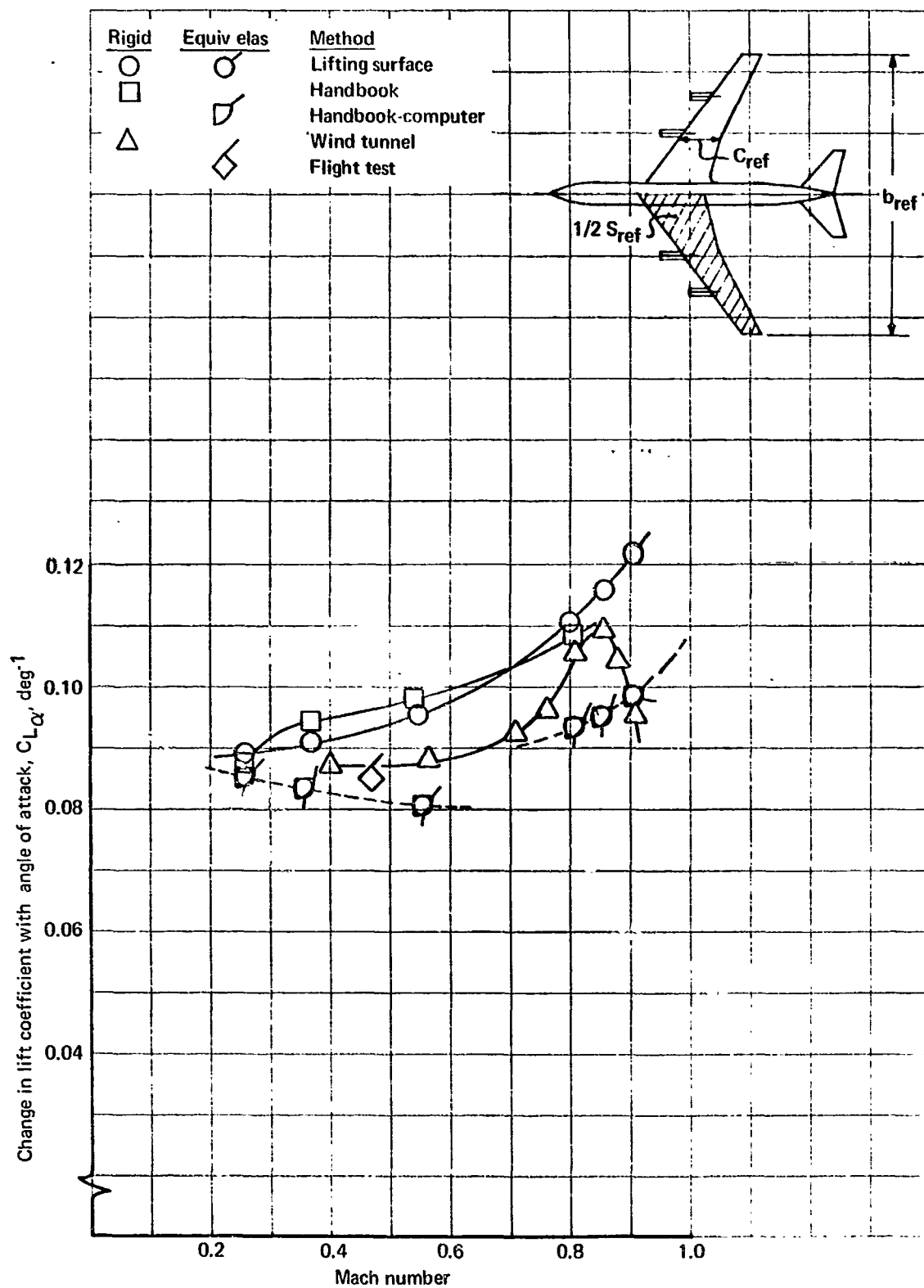


FIGURE 6.—COMPARISON OF ESTIMATION METHODS FOR  $C_{L_{\alpha}}$  -707-320B

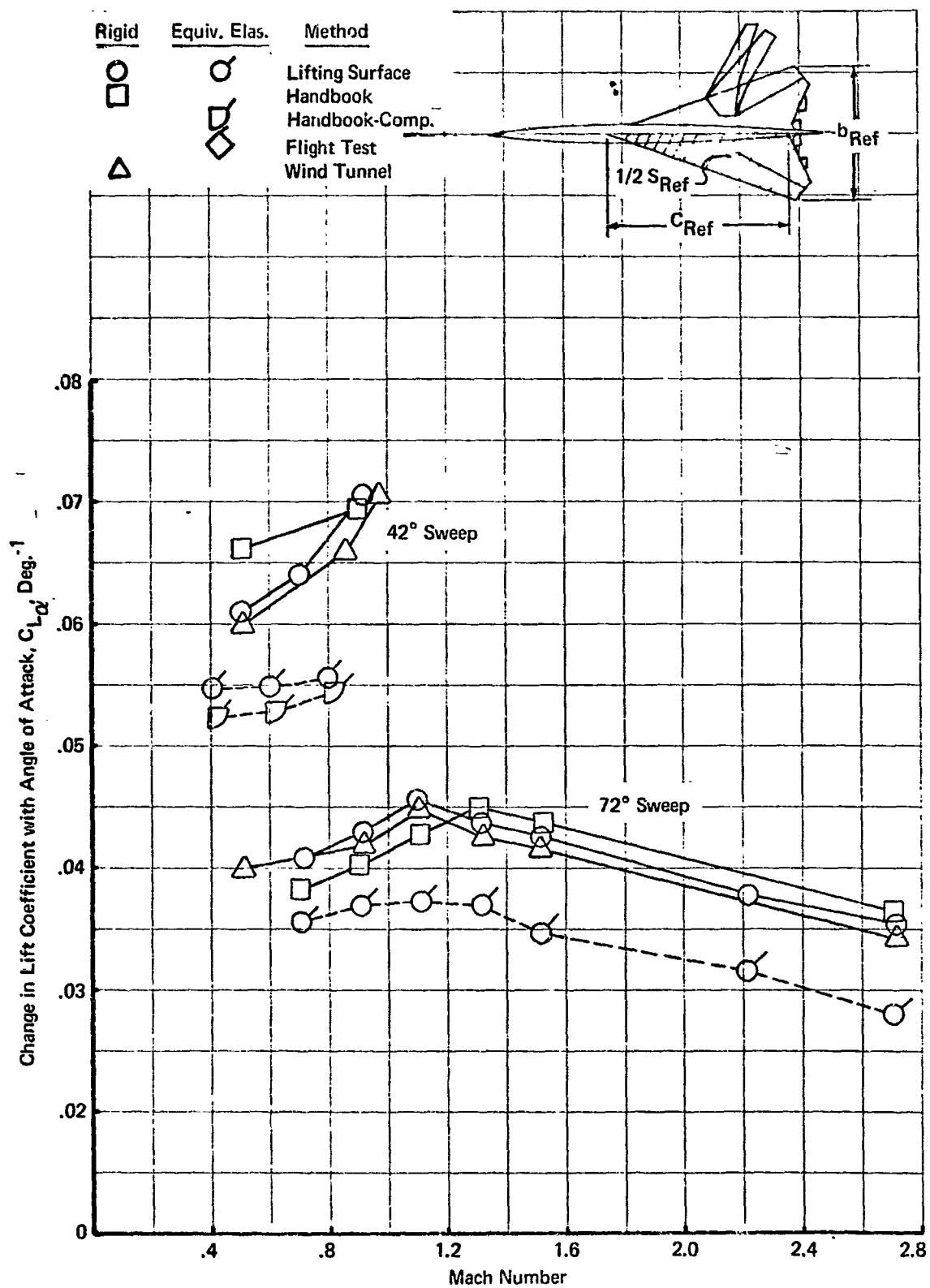


FIGURE 7. - COMPARISON OF ESTIMATION METHODS FOR  $C_{L\alpha}$ -SST

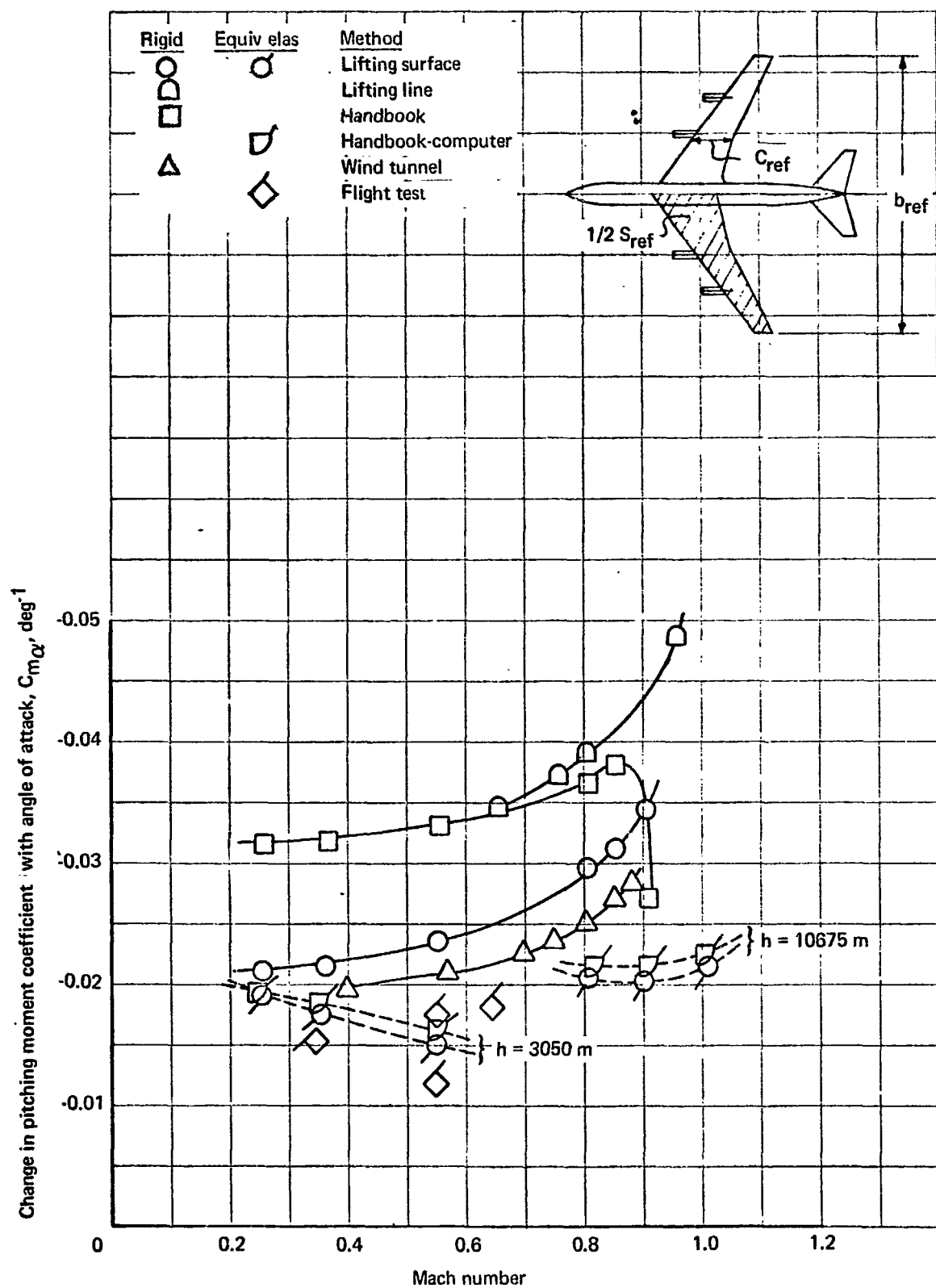


FIGURE 8. -COMPARISON OF ESTIMATION METHODS FOR  $C_{m\alpha}$  -707-320B

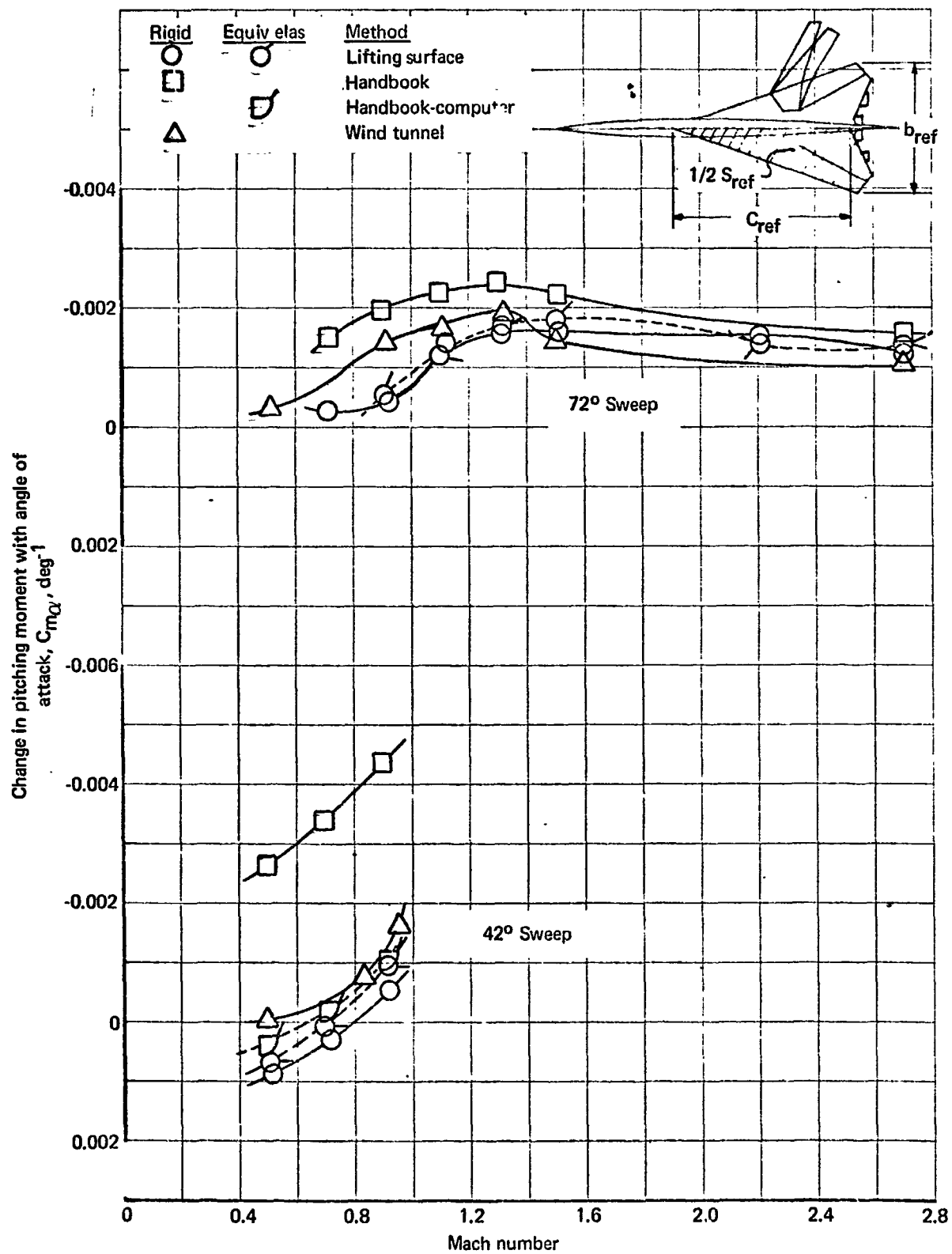


FIGURE 9. - COMPARISON OF ESTIMATION METHODS FOR  $C_{m\alpha}$  -SST

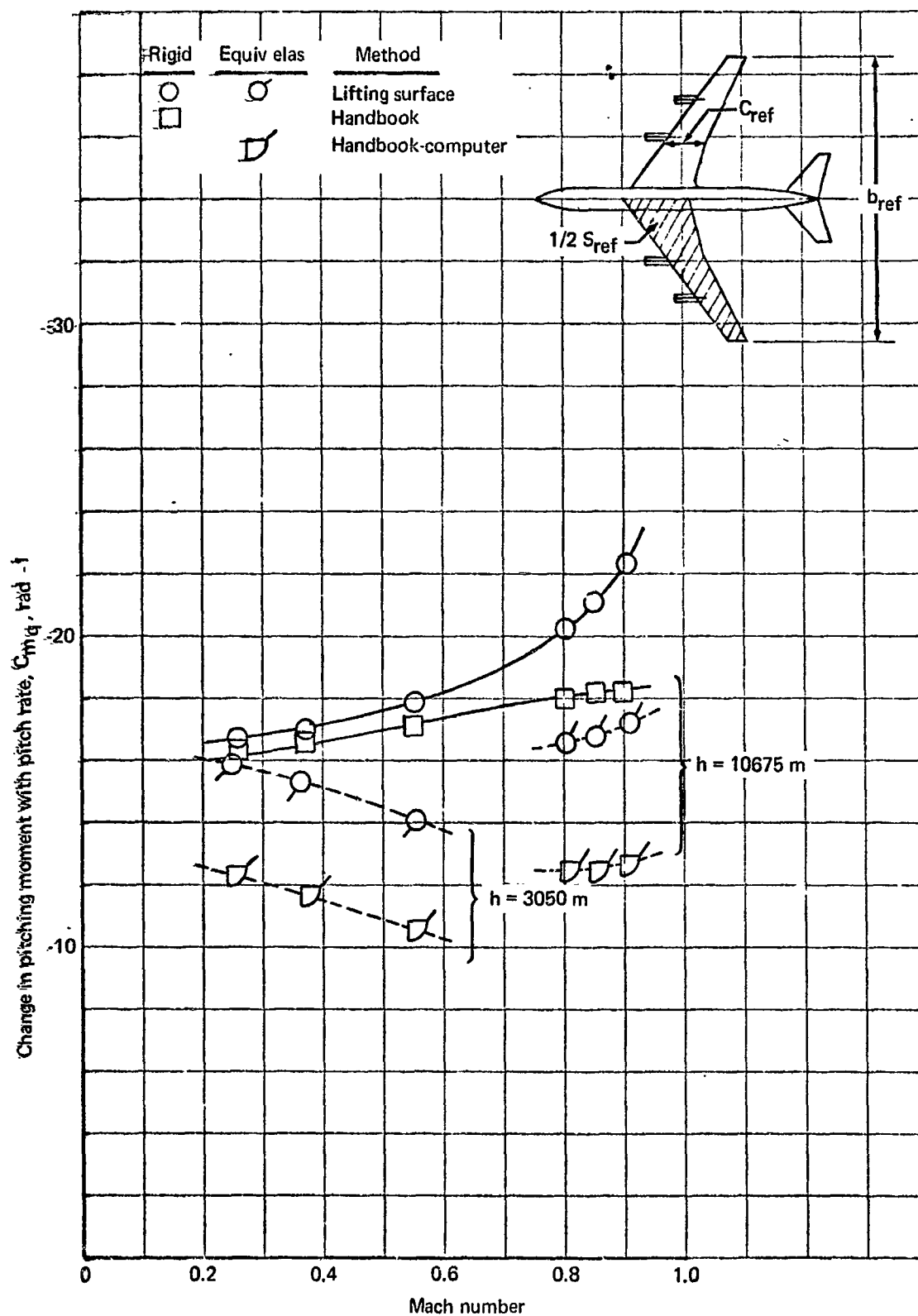


FIGURE 10. -COMPARISON OF ESTIMATION METHODS FOR  $C_{m_q}$  - 707-320B

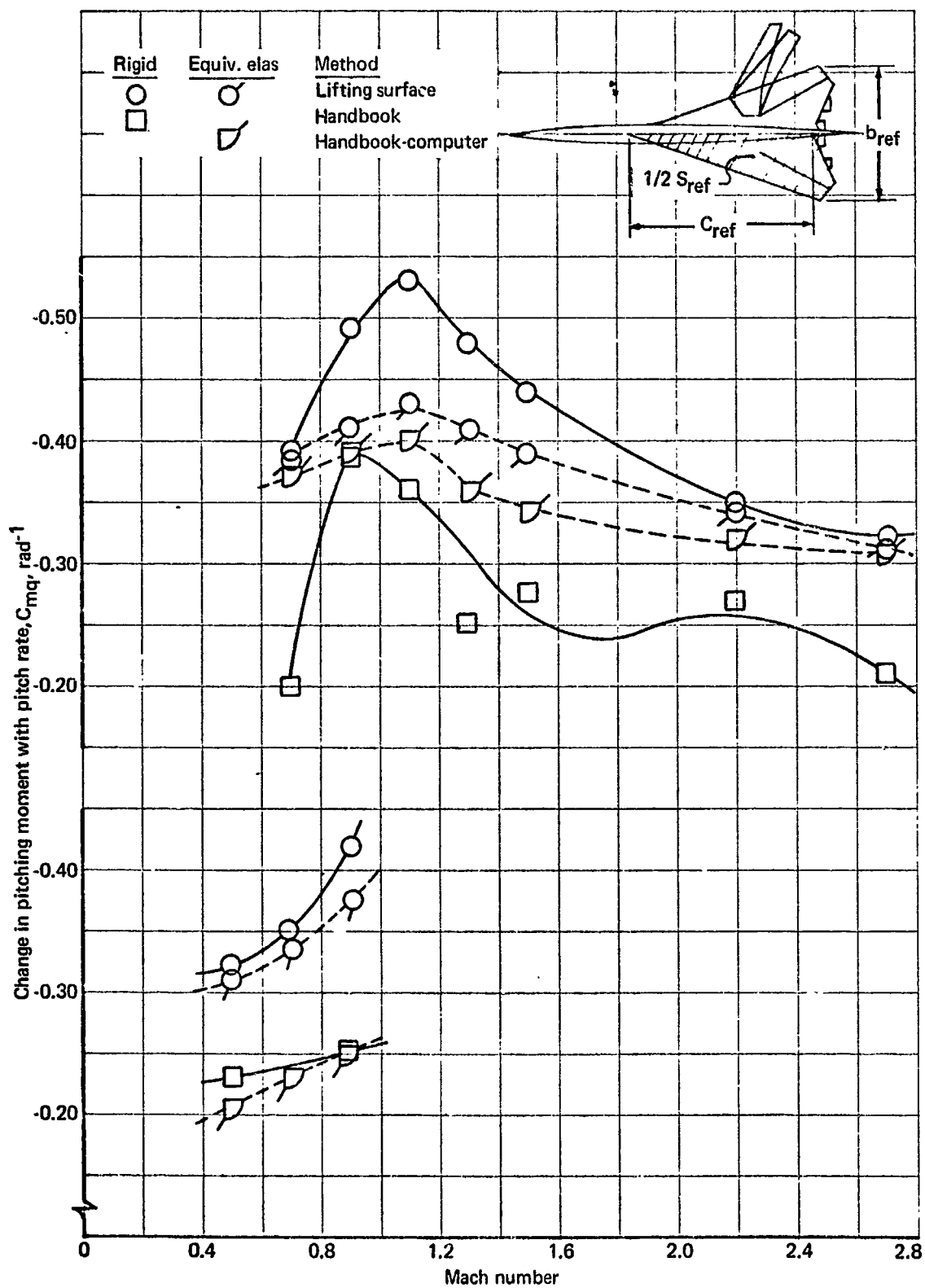


FIGURE 11. - COMPARISON OF ESTIMATION METHODS FOR  $C_{mq}$ -SST



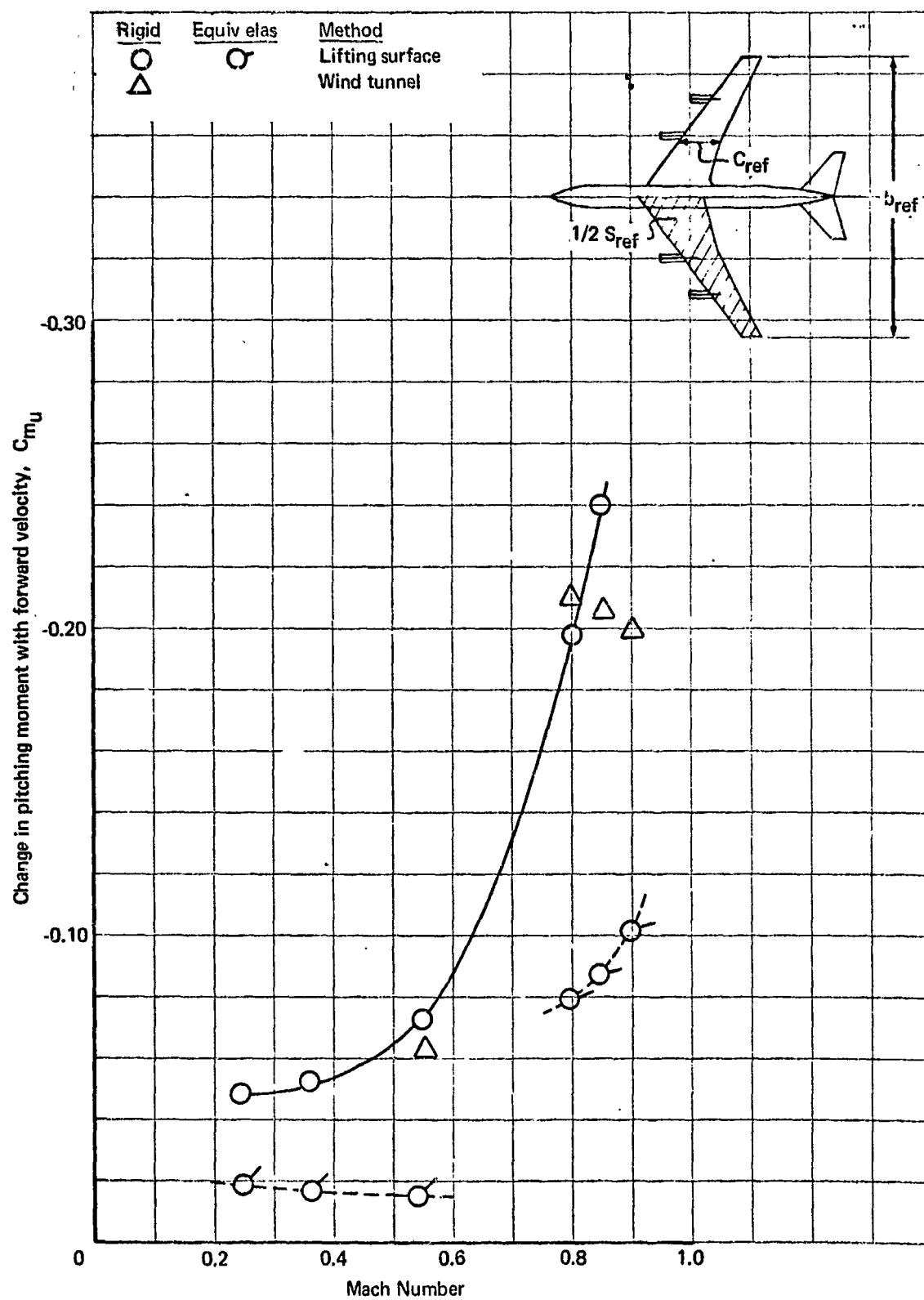


FIGURE 12. -COMPARISON OF ESTIMATION METHODS FOR  $C_{m_u}$  -707-320B

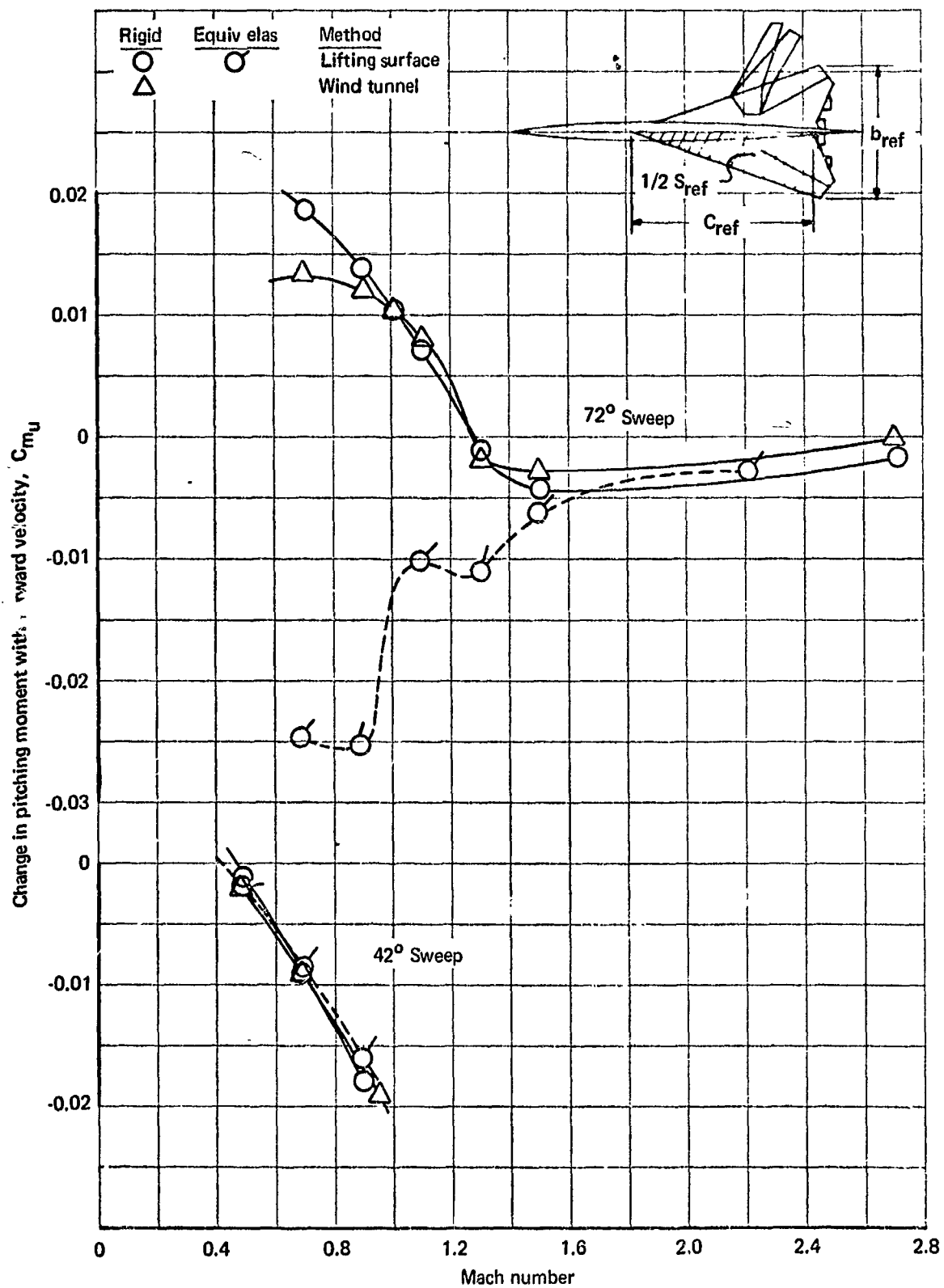


FIGURE 13. - COMPARISON OF ESTIMATION METHODS FOR  $C_{m_u}$  - SST

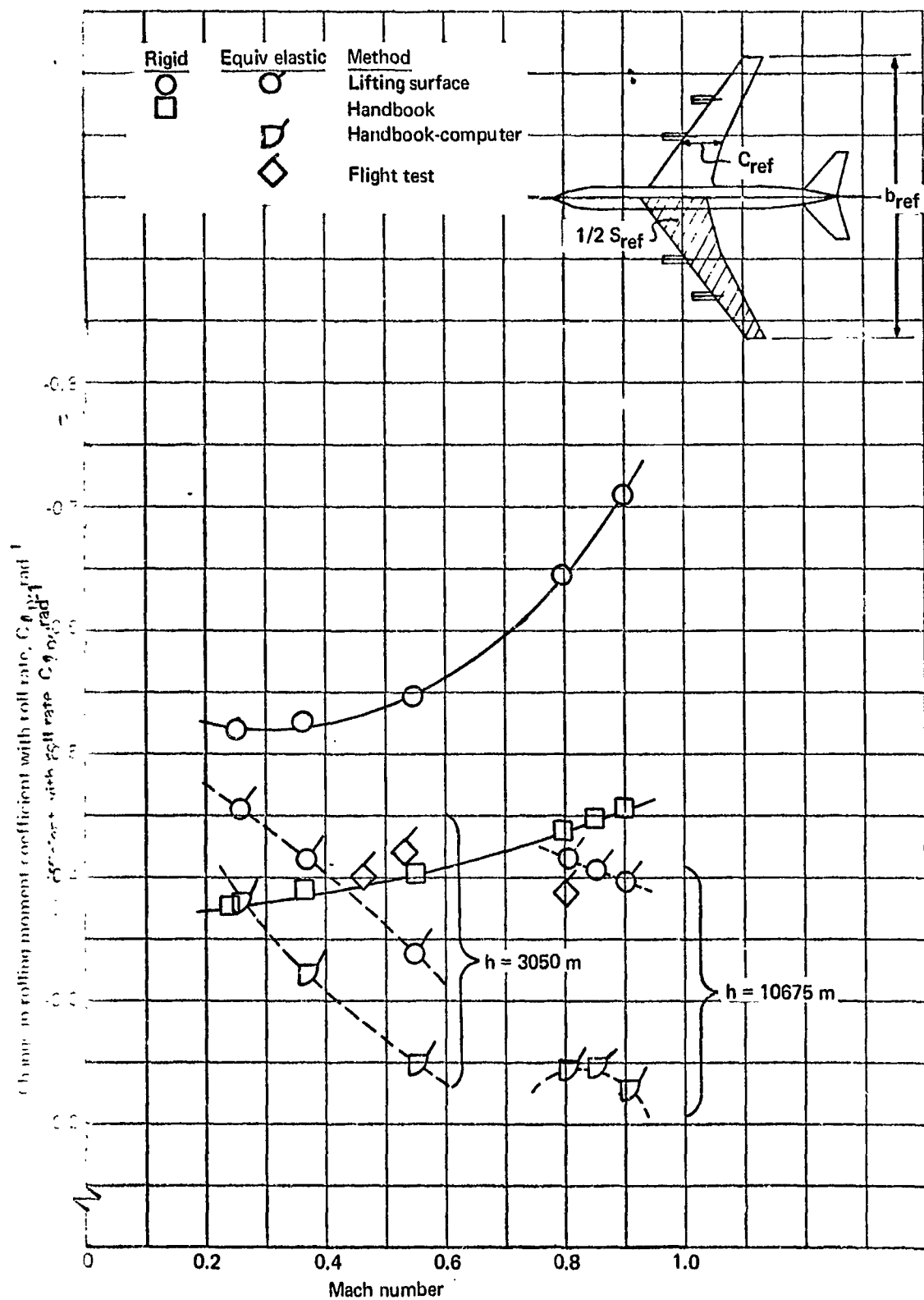


FIGURE 14. -COMPARISON OF ESTIMATION METHODS FOR  $C_{l_p}$  - 707-320B

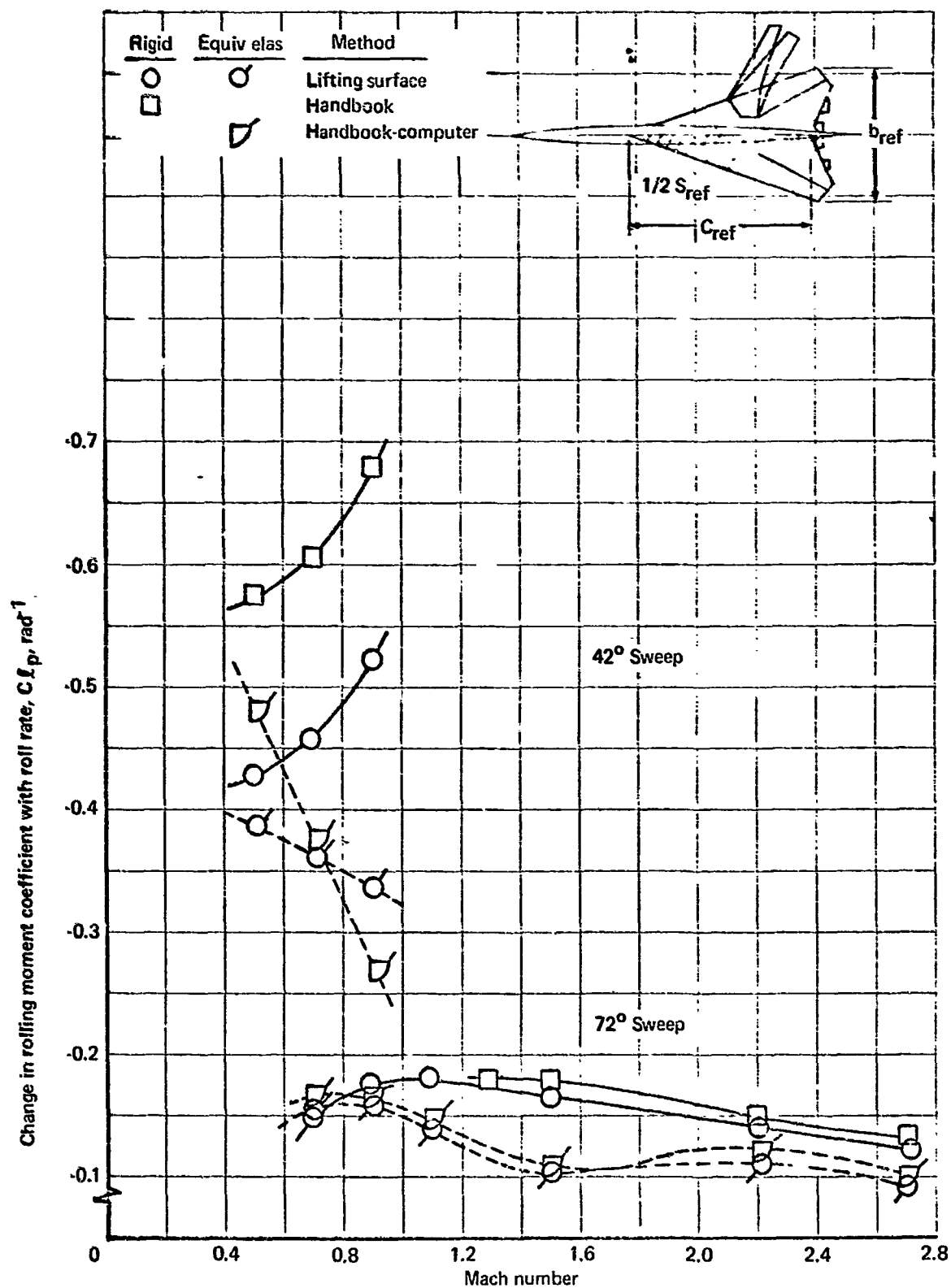


FIGURE 15. -COMPARISON OF ESTIMATION METHODS FOR  $C_{l_p}$  -SST

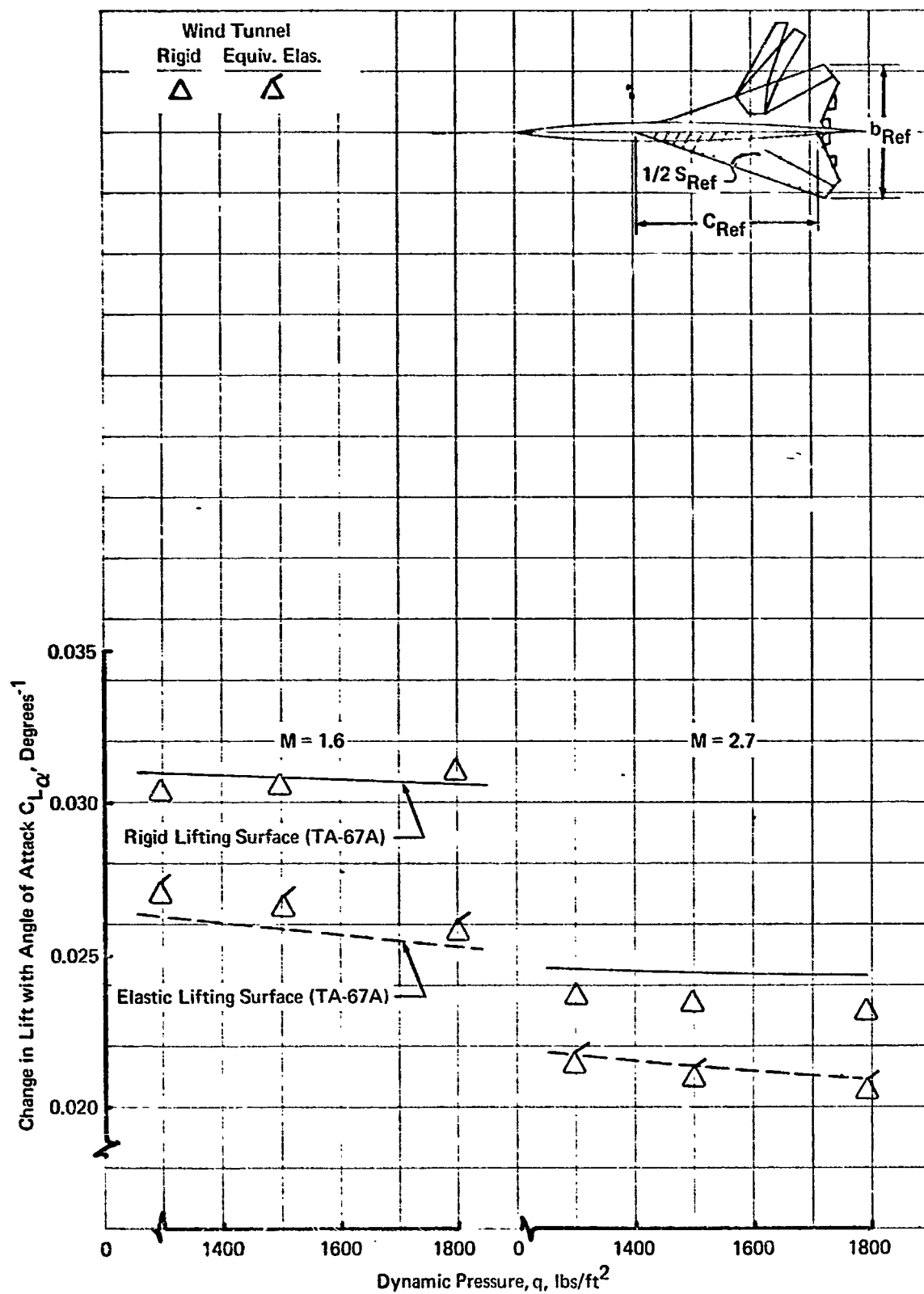


FIGURE 16. -EFFECT OF DYNAMIC PRESSURE ON  $C_{L\alpha}$ -SST

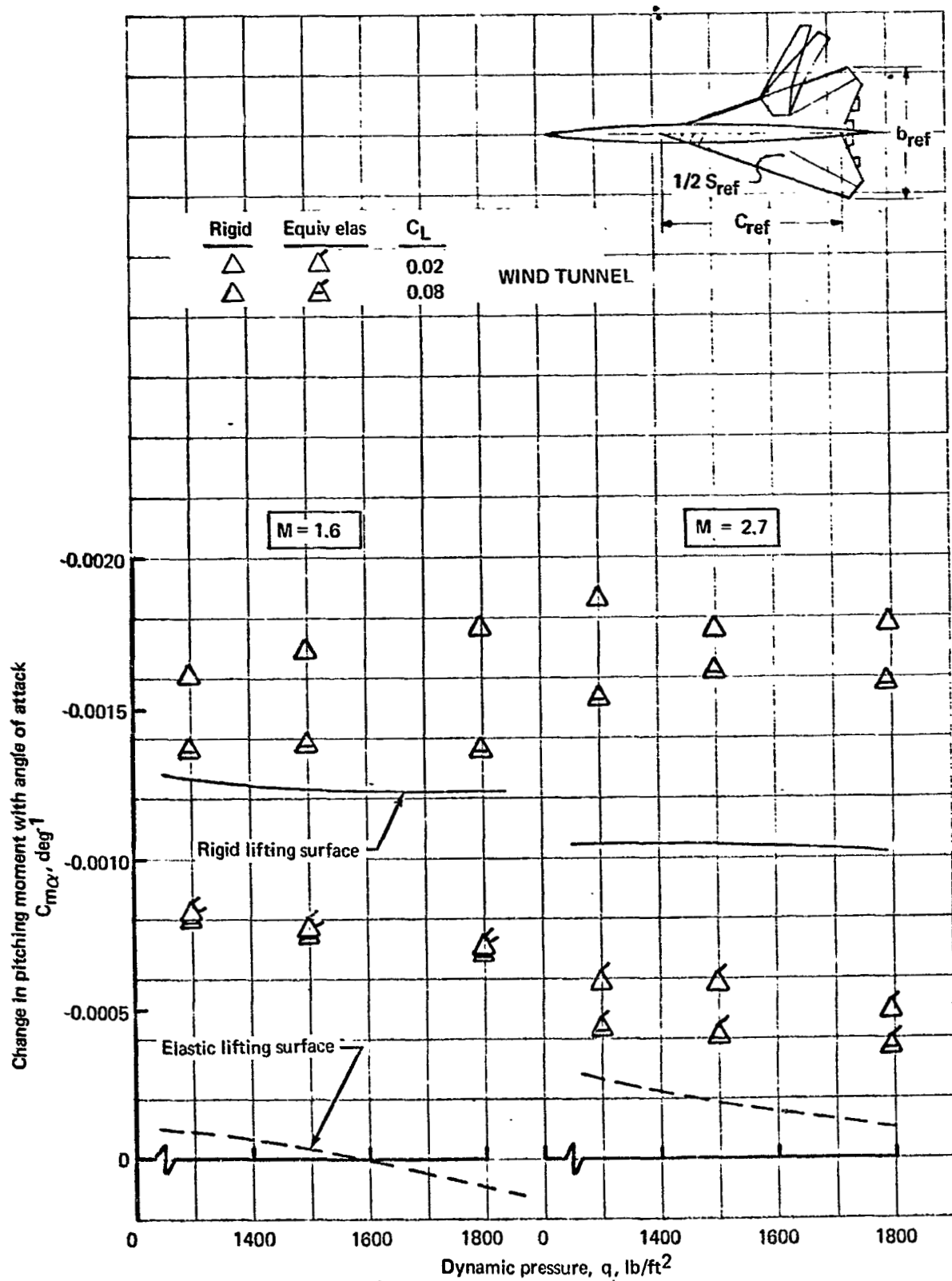


FIGURE 17. -EFFECT OF DYNAMIC PRESSURE ON  $C_{m\alpha}$ -SST

## **8.0 METHODS FOR CALCULATING STABILITY AND RESPONSE CHARACTERISTICS**

### **8.1 General Considerations**

The determination of static stability and control characteristics may be considered as an intermediate step in the solution of the total problem of dynamic stability and control. Taking this intermediate step has a number of advantages.

1. Steady-state trim, balancing, and control information are provided.
2. Comparison with static stability criteria can be made.
3. Through examining the static picture it is possible to predict some dynamic characteristics.

However, the analyst is limited in trying to predict the dynamic motion from static stability considerations for the following reasons.

1. Static longitudinal stability is, in general, a prerequisite for dynamic longitudinal stability. There are, however, instances where static stability is not required for dynamic stability.
2. Static stability is usually, but not necessarily, required for good handling qualities.
3. Structural dynamic motions cannot be investigated. Static stability analyses are only concerned with the rigid and equivalent elastic airplanes.

Static stability characteristics are found by an analysis of the stability derivatives (using calculation methods discussed in Sec. 7), and by solving for the neutral point, maneuver point, etc. through fairly elementary expressions containing the stability derivatives.

Dynamic stability characteristics are generally determined by analyzing the roots of the characteristic equation when the perturbations of the airplane motion from the steady state are small. Approximate formulas for the frequency and damping of the motion are at times applicable for small perturbation motion. When nonlinear effects are significant, time history solutions of the arbitrary or large perturbation equations are used to study the stability characteristics.

The three airplane models treated in this study, viz., rigid, equivalent elastic, and completely elastic airplanes, were derived from consideration of the degree of airplane flexibility. The completely elastic airplane has been analyzed with various elastic degrees of freedom, and the effects of residual flexibility have been studied. The various structural mathematical models have been reviewed in Sec. 5. Static and dynamic stability characteristics were determined by using stability derivatives from several sources: handbook formulas, aerodynamic lifting surface theory, and wind tunnel data.

General results obtained from the static and dynamic analyses are presented in tables 9 and 10. Table 9 summarizes the accuracies obtained from lifting surface theory and hand-book techniques in predicting the rigid airplane static stability and control characteristics when compared with wind tunnel predictions. A limited amount of substantiation with flight test data was obtained. Also shown in table 9 is the relative effect of elasticity on the various static characteristics. Although most of the substantiation of methods was for the rigid airplane, one could expect to obtain similar accuracies for the equivalent elastic airplane. A method based upon ratioing the elastic and rigid stability derivatives that can be expected to improve the prediction of static stability characteristics is discussed in par. 8.2. The poor accuracy obtained for some configurations and characteristics is almost entirely due to poor prediction of the stability derivative  $C_{m\alpha}$ . Appendix B discusses the calculation of this derivative and the expected improvements to the lifting surface theory mechanization program to improve the prediction of  $C_{m\alpha}$ .

Table 10 presents the same general results for the dynamic characteristics of the airplane studied. In addition, general results on the usefulness of approximate formulas for predicting frequency and damping are summarized, and the number of elastic modes needed for an accurate dynamic elastic analysis is given. Appendix C should be consulted for details of the derivatives that were used in an individual method and for a complete discussion of all the dynamic stability results. A limited discussion appears in par. 8.3.5 for some of the more important results.

## 8.2 Static Stability Characteristics

The static stability characteristics of an airplane are strongly dependent on the individual stability derivatives and on how the stability derivatives combine. The effects of the individual stability derivatives can be judged by comparing their signs with the static stability criteria, as discussed in Sec. 6, and by noting the magnitude of the derivative. The sign of the derivative simply indicates whether the airplane is stable, unstable, or neutrally stable with respect to a certain motion variable (requirements for stable motion are summarized in table 11). The magnitude of the derivative gives an indication of the degree of stability or instability.

The derivatives in combination can also be used to evaluate airplane stability characteristics. The static stability and control characteristics usually investigated are:

1. Elevator and stabilizer trim angles;
2. Stick-speed stability;
3. Elevator and stabilizer angles per g;
4. Neutral point;
5. Maneuver point.

Expressions used for calculating the above quantities are given in equations (154) through (161).



**TABLE 9.— STATIC STABILITY CALCULATIONS-GENERAL RESULTS**

Stability and control characteristic	Relative accuracy of calculation method <sup>a</sup>							Relative effect of elasticity <sup>b</sup>		
	Computer lifting surface, rigid and equivalent elastic				USAF handbook					
	Sub		Super		Sub		Super	Sub		Super
	707	SST	SST	707	SST	SST		707	SST	SST
	Rigid	E.E.	Rigid	Rigid	Rigid	Rigid	Rigid			
$\delta E_1, iH_1$	G	c	P	F	P	P	G	L	L	L
$d\delta_E/dV$	G	G <sup>d</sup>	P	F	P	P	G	M	S	M
$d\delta_E/dn$	G	G <sup>d</sup>	P	G	P	-	F	S	M	M
$h_n$	G	c	P	F	P	P	P	L	M	M
$h_m$	G	c	P	F	P	P	F	L	M	M

- a. Reflects almost entirely ability to calculate derivative  $C_{m\alpha}$  and resulting effect on characteristic.  
 G (good)—method compares favorably with wind tunnel predictions (exception allowed.)  
 F (fair)—less favorable correlation with predictions.  
 P (poor)—method does not compare favorably with predictions.
- b. L (large)—elasticity considered a significant effect.  
 M (moderate)—elasticity considered moderately important; not quite as significant as differences due to stability derivative calculation methods.  
 S (small)—elasticity considered a minor change to stability characteristic; changes due to stability derivative calculation methods usually much more important.
- c. No data available.
- d. Correlation with flight test, but based on a very limited amount of data.

TABLE 10. - DYNAMIC STABILITY CALCULATIONS-GENERAL RESULTS

	Relative accuracy of calculation method <sup>a</sup>										Relative effect of elasticity <sup>b</sup>			Approx. no. of elastic modes for accurate comp. elastic results		
	Computer lifting surface theory for long. modes; NACA TR1098 for lat.-dir. modes <sup>c</sup>				Handbook quartic solution				Approx. eqs. (accuracy or usefulness) <sup>d</sup>							
	Sub		Super		Sub		Super		Sub		Super		Sub		Super	
	707 Rigid	SST Rigid	SST Rigid	SST Rigid	707 Rigid	E.E. Rigid	SST Rigid	SST Rigid	707 SST	SST	707 SST	SST	707 SST	SST	707 SST	SST
Longitudinal																
Short period: frequency damping	G	P	G	G	G	e.	P	G	G	G	G	S	S	10	20	15
	G	P	G	G	G	e.	F	F	G	G	G	S	M	10	15	15
Phugoid mode: frequency damping	F	P	G	F	P	e.	P	P	P	P	P	M	S	L	Not done	
	F	P	F	P	P	e.	P	P	P	P	P	M	M	L	Not done	
Lateral-directional																
Dutch roll: frequency damping	G	G	No	F	P <sup>f</sup>	F	F	F	F	F	F	S	S	15	?	?
	F	F	method	P	P <sup>f</sup>	F	F	F	F	F	P	M	L	12	?	?
Roll mode: damping	F	G	No	F	e.	G	G	G	Not done <sup>g</sup>			S	S	S	Not done	
Spiral mode: damping	F	G	method	P	e.	F	F	P	P	P	M	M	S			done

- a. G(good)—method compares favorably with wind tunnel predictions (exception allowed).  
 F(fair)—less favorable correlation with predictions.  
 P(poor)—method does not compare favorably with predictions.
- b. L(large)—elasticity considered a significant effect.  
 M(moderate)—elasticity considered moderately important; not quite as significant as differences due to stability derivative calculation methods.  
 S(small)—elasticity considered a minor change to stability characteristic; changes due to stability derivative calculation methods usually much more important.
- c. No lifting surface theory method mechanized for lateral-directional derivatives.  
 d. Compared to rigid and equivalent elastic quartic solution.  
 e. No data available.  
 f. Based on comparison with flight test data.  
 g. Quartic solution more efficient.

TABLE 11. – STATIC STABILITY CRITERIA

Observation (Section 6)	
$C_{D_u} > 0$	$C_{l_p} < 0$
$C_{Y_\beta} < 0$	$C_{m_q} < 0$
$C_{L_\alpha} > 0$	$C_{n_r} < 0$
$C_{n_\beta} > 0$	$C_{m_u} > 0$
$C_{m_\alpha} < 0$	$C_{l_\beta} < 0$
Calculation	
Stick-speed stability (Section 6)	$d\delta_{E_1}/dV > 0$
Elevator angle per g	$d\delta_{E_1}/dn _{V_{c_1}} < 0$
Neutral point (p. 78, ref. 4)	Aft of aft c.g. limit
Maneuver point (p. 59, ref. 4)	Aft of aft c.g. limit

8.2.1 Elevator and stabilizer trim angles.—For rigid and equivalent elastic airplanes, the horizontal stabilizer angle to trim is given by

$$(D12) \quad i_{H_1} = \frac{C_{L_\alpha} C_{m_0} + C_{m_\alpha} (C_{L_1} - C_{L_0})}{C_{m_\alpha} C_{L_{i_H}} - C_{m_{i_H}} C_{L_\alpha}} \quad (154)$$

Similarly, the elevator angle to trim is given by

$$\delta_{E_1} = \frac{C_{L_\alpha} C_{m_0} + C_{m_\alpha} (C_{L_1} - C_{L_0})}{C_{m_\alpha} C_{L_{\delta_E}} - C_{m_{\delta_E}} C_{L_\alpha}} \quad (155)$$

8.2.2 Stick-speed stability.—The elastic airplane stick-speed stability for stabilizer trim is given by the following expression:

$$\begin{aligned} \frac{d(i_H)}{dV} = & \frac{1}{C_{m_\alpha} C_{L_{i_H}} - C_{m_{i_H}} C_{L_\alpha}} \left[ \left\{ C_{L_\alpha} \frac{\partial C_{m_0}}{\partial \bar{q}} + C_{m_0} \frac{\partial C_{L_\alpha}}{\partial \bar{q}} \right. \right. \\ & - C_{m_\alpha} \left( \frac{C_{L_1}}{\bar{q}} + \frac{\partial C_{L_0}}{\partial \bar{q}} \right) + (C_{L_1} - C_{L_0}) \frac{\partial C_{m_\alpha}}{\partial \bar{q}} - i_{H_1} \left( C_{m_\alpha} \frac{\partial C_{L_{i_H}}}{\partial \bar{q}} \right. \\ & \left. \left. + C_{L_{i_H}} \frac{\partial C_{m_\alpha}}{\partial \bar{q}} - C_{m_{i_H}} \frac{\partial C_{L_\alpha}}{\partial \bar{q}} - C_{L_\alpha} \frac{\partial C_{m_{i_H}}}{\partial \bar{q}} \right) \right] \rho V C_1 \\ & + \left\{ \frac{\partial C_{m_\alpha}}{\partial M} (C_{L_1} - C_{L_0}) - C_{m_\alpha} \frac{\partial C_{L_0}}{\partial M} + C_{L_\alpha} \frac{\partial C_{m_0}}{\partial M} + C_{m_0} \frac{\partial C_{L_\alpha}}{\partial M} \right. \\ & - i_{H_1} \left( \frac{\partial C_{m_\alpha}}{\partial M} C_{L_{i_H}} + C_{m_\alpha} \frac{\partial C_{L_{i_H}}}{\partial M} - C_{L_\alpha} \frac{\partial C_{m_{i_H}}}{\partial M} \right. \\ & \left. \left. - C_{m_{i_H}} \frac{\partial C_{L_\alpha}}{\partial M} \right) \right\} \left[ \frac{1}{a} \right] \quad (156) \end{aligned}$$

For the rigid airplane all variations with dynamic pressure vanish, i.e.,  $\partial C_{m_0}/\partial \bar{q} = \partial C_{L_\alpha}/\partial \bar{q} = 0$ , etc. Therefore, for the rigid airplane equation (156) becomes

$$\begin{aligned} \frac{d(i_H)}{dV} = & \frac{1}{C_{m_\alpha} C_{L_{i_H}} - C_{m_{i_H}} C_{L_\alpha}} \left[ -C_{m_\alpha} \frac{C_{L_1}}{\bar{q}} \rho V C_1 + \left( \frac{\partial C_{m_\alpha}}{\partial M} (C_{L_1} - C_{L_0}) \right. \right. \\ & - C_{m_\alpha} \frac{\partial C_{L_0}}{\partial M} + C_{L_\alpha} \frac{\partial C_{m_0}}{\partial M} + C_{m_0} \frac{\partial C_{L_\alpha}}{\partial M} - i_{H_1} \left( \frac{\partial C_{m_\alpha}}{\partial M} C_{L_{i_H}} \right. \\ & \left. \left. + C_{m_\alpha} \frac{\partial C_{L_{i_H}}}{\partial M} - C_{L_\alpha} \frac{\partial C_{m_{i_H}}}{\partial M} - C_{m_{i_H}} \frac{\partial C_{L_\alpha}}{\partial M} \right) \right] \frac{1}{a} \end{aligned} \quad (157)$$

As shown in table 11, a stable gradient of longitudinal control displacement versus speed is defined as one for which  $di_H/dV|_{n=1} > 0$ . Stick-speed stability is usually referred to as a handling-qualities parameter. The equations for elevator speed stability are, of course, identical with equations (156) and (157), except that  $\delta_E$  replaces  $i_H$  in all terms.

**8.2.3 Elevator angle per g.**—An expression for elevator angle per  $g$  in a normal pullup can be written as

$$\frac{d\delta_E}{dn} = - \frac{C_{m_\alpha} C_{L_1} + \frac{g\bar{c}}{2V^2} (C_{L_\alpha} C_{m_q} - C_{m_\alpha} C_{L_q})}{C_{L_\alpha} C_{m_{\delta_E}} - C_{m_\alpha} C_{L_{\delta_E}}} \quad (158) \quad (D3)$$

Equation (158) holds for both rigid and equivalent elastic airplanes. A stable gradient of elevator displacement versus load factor is defined as one that satisfies

$$\left. \frac{d\delta_E}{dn} \right|_{V=c_1} < 0 \quad (159)$$

The term containing the stability derivative  $C_{L_q}$  is usually insignificant.

**8.2.4 Neutral point.**—The neutral point is calculated from the expression

$$h_n = h - \frac{C_{m_\alpha}}{C_{L_\alpha}} \quad (160)$$

where  $h$  is the center-of-gravity position. The requirement on the neutral point is that it should be aft of the aft center-of-gravity limit. Equation (160) is good only for small angles of attack in the linear range.

**8.2.5 Maneuver point.**—An expression to calculate the maneuver point is

$$h_m = h_n - \frac{C_{m_q}}{\frac{4m}{\rho S \bar{c}} - C_{L_q}} \quad (161)$$

The effect of the derivatives  $C_{L_q}$  on the maneuver point is negligible, as it was for the elevator-angle-per-g characteristic. The requirement on the maneuver point is that it should be aft of the aft center-of-gravity limit.

Complete derivations of the above equations for the static stability characteristics can be found in app. C.

Five methods were used in this study to determine the longitudinal derivatives.

	Applicability	
	Rigid Airplane	Equivalent Elastic Airplane
1. Computer, using lifting surface theory (aerodynamic influence coefficient method)	yes	yes
2. Handbook	yes	no
3. Handbook combined with computer lifting surface theory	no	yes
4. Wind tunnel	yes	yes (limited)
5. Flight test	no	yes (limited)

The characteristics of the rigid airplane were compared using the previously listed expressions for static stability and control.

Elastic stability characteristics were determined using equivalent elastic stability derivatives of the Formulation II type (app. B). These derivatives were generated by methods 1. and 3. above.

Some flight test data were available to correlate theoretical predictions of longitudinal static stability. The data were obtained as part of the certification requirements for longitudinal stability and control for the model 707-320B airplane. Results for both maneuvering stability,  $d\delta_E/dn$  and stick-speed stability,  $d\delta_E/dV$ , were obtained from these tests. The elevator-angle-per-g information was reduced from windup turn maneuvers.

Figure 18 illustrates some typical static margin results for the 707-320B and the 42°-wing-sweep SST configuration. Variations in the results due to both stability derivative calculation technique and elasticity can be noted. Appendix C contains detailed results for all characteristics in graphic form.

Lifting surface theory (aerodynamic influence coefficient method) was found from the static stability analysis to give better predictions than handbook methods for some cases (fig. 18 and table 9). It gives direct, acceptable results for some stability characteristics for equivalent elastic and rigid airplanes. However, if wind tunnel data are available a more accurate way of predicting elastic effects would be to compute an elastic-to-rigid ratio or increment referenced to the wind tunnel value. This was substantiated in Sec. 7. For example,

$$C_{m_{\alpha}}^{\text{EqEl}} = C_{m_{\alpha}}^{\text{WT}} \left( \frac{C_{m_{\alpha}}^{\text{Elastic}}}{C_{m_{\alpha}}^{\text{Rigid}}} \right)_{\text{Computed}}$$

### 8.3 An Analysis of Dynamic Stability by Characteristic Equation Rooting

**8.3.1 Applicability of characteristic equation methods.**— Airplane equations of motion can be reduced to a set of linear, second-order differential equations with constant coefficients when dynamic behavior can be approximated by assuming that motion perturbations relative to the steady state are small (see Sec. 5). These equations are called small perturbation equations of motion and are amenable to generating characteristic equations whose roots can be examined to determine motion characteristics.

The dynamic stability criteria for all characteristic equation roots are given in par. 6.2.1. These criteria can be satisfied by inspection, i.e., by checking the sign of the real parts of the roots or their absence. However, these yes/no-type answers relate very little information about airplane motion characteristics. Some of the parameters that can be deduced from the roots and are more physically oriented are discussed in par. 8.3.2.

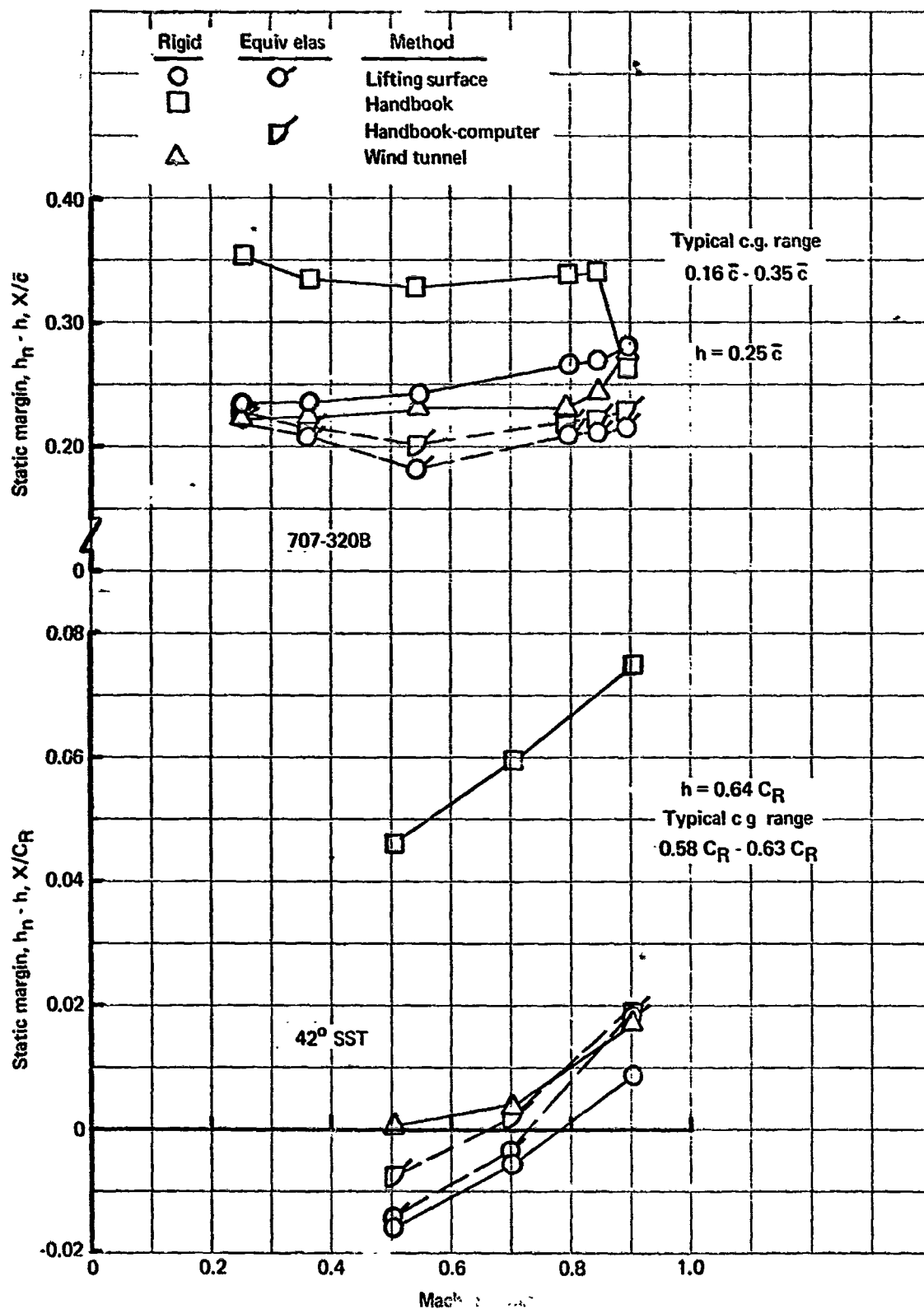


FIGURE 18.— 707-320B and 42° SST STATIC MARGIN, RIGID VS. EQUIVALENT ELASTIC



**8.3.2 Rigid and equivalent elastic airplanes.**— The small perturbation longitudinal and lateral-directional equations of motion are given as equations (17) and (18). It is possible to take the Laplace transformation of the equations of motion and solve for the roots of the resulting characteristic equations. However, the program that generated the roots and associated data in this study used a different technique, which will now be described briefly.

The lateral-directional characteristic equation, with the indicated rigid/equivalent elastic definitions, evolves from equations (17) and (18) as indicated in the equation in table 12. This equation is a result of the requirement that (for  $\delta_A = \delta_R = 0$ ) the equation

$$[A(D)] \{x\} = \{0\} \quad (162a)$$

has nontrivial solutions, i.e.,

$$\{x\} = \begin{Bmatrix} \beta \\ \phi \\ \hat{r} \end{Bmatrix} \neq \{0\} \quad (162b)$$

The longitudinal characteristic equation in determinant form evolves as indicated in the equation in table 13. Inertia relief is handled implicitly as indicated in table 13 (called Formulation II in app. B). The effects due to  $\ddot{\theta}$  cannot be treated explicitly and are handled as summarized in table 13.

The expanded form of the characteristic equations of tables 12 and 13 for the small perturbation equations of motion written in the parameter  $\lambda$  is

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \quad (163)$$

where the coefficients A through E are determined by the case (longitudinal or lateral-directional). Equation (163) is obtained by assuming the solutions

$$\text{longitudinal mode} \quad \begin{cases} \hat{u} = \hat{u}_0 e^{\lambda t/t^*} \\ \alpha = \alpha_0 e^{\lambda t/t^*} \\ \Theta = \Theta_0 e^{\lambda t/t^*} \end{cases} \quad (164)$$

or

$$\text{lateral-directional mode} \quad \begin{cases} \beta = \beta_0 e^{\lambda t/t^*} \\ \phi = \phi_0 e^{\lambda t/t^*} \\ \hat{r} = \hat{r}_0 e^{\lambda t/t^*} \end{cases} \quad (165)$$

and substituting these into the equations in the form of equation (162a). After carrying out the differentiation,  $e^{\lambda t/t^*}$  can be eliminated, leaving

TABLE 12.— LATERAL-DIRECTIONAL CHARACTERISTIC EQUATION IN DETERMINANT FORM

$\begin{vmatrix} (2\mu - C_{y\beta}) D - C_{y\beta} & -(C_{yp} D + C_{LI}) & (2\mu - C_{yr}) \\ -(C_{l\beta} + C_{l\beta}^* D) & i_A D^2 - C_{lp} D & -(i_E D + C_{lr}) \\ -(C_{n\beta} + C_{n\beta}^* D) & -(i_E D^2 + C_{np} D) & i_c D - C_{nr} \end{vmatrix} = 0$ <p>where: <math>\mu = M/\rho S_w \ell</math>; <math>M = W/g</math>; <math>\ell = b/2</math>  <math>D = t^* d/dt</math>; <math>t^* = \ell/V_{c1}</math></p> <p>Note: For the equivalent elastic airplane, <math>C_{ypI}</math>, <math>C_{yrI}</math>, <math>C_{l\beta I}</math>, and <math>C_{n\beta I}</math> are not accounted for.</p>	
<p>Rigid airplane (S8)</p>	<p>Equivalent plastic airplane (S3) &amp; (S7)</p>
<p>All derivatives are conventional rigid derivatives.</p> <p><math>i_A = I_{xx}/\rho S_w \ell^3</math>  <math>i_C = I_{zz}/\rho S_w \ell^3</math>  <math>i_E = I_{xz}/\rho S_w \ell^3</math></p> <p><math>W = \text{Actual Gross Weight} - W_I</math></p>	<p>Derivatives include elastic effects due to aerodynamic loading (Formulation I) and would have E subscripts.</p> <p><math>i_A = (I_{xx} - C_{lpI} \bar{q} S_w b)/\rho S_w \ell^3</math>  <math>i_C = (I_{zz} - C_{nrI} \bar{q} S_w b)/\rho S_w \ell^3</math>  <math>i_E = \frac{(I_{xz} + \frac{1}{2} C_{lrI} \bar{q} S_w b + \frac{1}{2} C_{npI} \bar{q} S_w b)}{\rho S_w \ell^3}</math></p> <p><math>W = g(M - C_{y\ddot{y}I} \bar{q} S_w)</math>; <math>M = W_I/g</math></p>
<p>Assumptions required:</p> <p>D1 and 2, D5, D7, D11, G1-6, G10, S1 and 2, S5 and R1 = <math>\phi_I = \beta_I = 0</math></p>	

TABLE 13.- LONGITUDINAL CHARACTERISTIC EQUATION IN DETERMINANT FORM

$\begin{vmatrix} 2\mu D + C_{D_u} + 2C_{D_1} & C_{D_\alpha} - C_{L_1} & C_{L_1} \\ C_{L_u} + 2C_{L_1} & (2\mu + C_{L_\alpha})D + (C_{L_\alpha} + C_{D_1}) & -(2\mu - C_{L_q})D \\ -C_{m_u} & -(C_{m_\alpha}D + C_{m_\alpha}) & i_B D^2 - C_{m_q}D \end{vmatrix} = 0$ <p>where: <math>\mu = M/\rho S_w \ell</math>; <math>M = W/g</math>; <math>\ell = \bar{c}/2</math></p> <p><math>D = t^* d/dt</math>; <math>t^* = \ell/V_{c_1}</math></p> <p>Note: For the equivalent elastic airplane, <math>C_{L\ddot{\theta}_I}</math> and all <math>\dot{u}_I</math> are not accounted for.</p>	
Rigid airplane (S8)	Equivalent elastic airplane (S3) & (S7)
<p>All derivatives are conventional rigid derivatives.</p> <p><math>i_B = I_{yy}/\rho S_w \ell^3</math></p> <p>W = Actual airplane weight</p>	<p>Derivatives include elastic effects due to both aerodynamic and inertial loading (Formulation II) and would have E subscripts.</p> <p><math>i_B = (I_{yy} = C_{m\ddot{\theta}_I} \bar{q} S_w \bar{c})/\rho S_w \ell^3</math></p> <p>W = Actual airplane weight</p>
<p>Assumptions required:</p> <p>D1-2, 5, 7, 8, 11, G1-6, 10, S1-2,5 and <math>R_1 = \phi_1 = \beta_1 = C_{T_{m_1}} + C_{m_1} = 0</math></p>	

$$[A(\lambda)] \{x_0\} = \{0\} \quad (166)$$

where  $[A(\lambda)]$  is identical with  $[A(D)]$  for  $\lambda = D$ . The solutions  $\lambda_i/t^*$ ,  $i = 1, 2, 3, 4$  of the fourth-order polynomial equation (163) are the "roots of the characteristic equation." Since a rather complete discussion of the coefficients A through E and the occurrence and significance of various combinations of real and complex roots may be found in refs. 4 and 36, it is not repeated here.

The systems analysis analogies used for airplane characteristic equations lead to the conventional "mode" definitions which follow (in equation (163) let  $\lambda_i/t^* = S$  and divide by  $A_4$ ).

First, for longitudinal equations (for two complex pairs of roots):

$$(S^2 + 2 \zeta_p \omega_{n_p} S + \omega_{n_p}^2) (S^2 + 2 \zeta_{sp} \omega_{n_{sp}} S + \omega_{n_{sp}}^2) = 0$$

where:  $p \sim$  phugoid mode

$sp \sim$  short period mode

$$\omega_{n_p} < \omega_{n_{sp}}$$

the roots are  $S_{1,2} = \sigma_p \pm j\omega_p$

$$S_{3,4} = \sigma_{sp} \pm j\omega_{sp}$$

$-\zeta_{p(sp)} \omega_{n_{p(sp)}} = \sigma_{p(sp)}$  is the real part of the phugoid (short period) root pair

$\omega_{n_{p(sp)}} \sqrt{1 - \zeta_{p(sp)}^2} = \omega_{p(sp)}$  is the imaginary part of the phugoid (short period) root pair

$\zeta_{p(sp)}$  is the phugoid (short period) damping ratio

$\omega_{n_{p(sp)}}$  is the phugoid (short period) undamped natural frequency

Second, for lateral-directional equations (for one complex pair and two real roots):

$$(S^2 + 2 \zeta_D \omega_{nD} S + \omega_{nD}^2) (S + \frac{1}{T_r}) (S + \frac{1}{T_s})^* = 0$$

where:  $D \sim$  Dutch roll mode

$r \sim$  rolling convergence root

$s \sim$  spiral root

$$\frac{1}{T_r} > \frac{1}{T_s}$$

the roots are  $S_{1,2} = \sigma_D \pm j \omega_D$

$$S_{3,4} = -\frac{1}{T_r}, -\frac{1}{T_s} = \sigma_3, \sigma_4$$

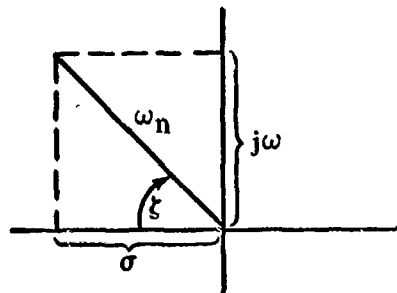
$-\zeta_D \omega_{nD} = \sigma_D$  is the real part of the Dutch roll root pair

$\omega_{nD} \sqrt{1 - \zeta_D^2} = \omega_D$  is the imaginary part of the Dutch roll root pair

$\zeta_D$  is the Dutch roll damping ratio

$\omega_{nD}$  is the Dutch roll undamped natural frequency

The relationship between the damping factor ( $\zeta$ ), undamped natural frequency ( $\omega_n$ ), damping ( $\sigma$ ) frequency, and damped frequency ( $\omega$ ) is illustrated in the diagram below.



\*Sometimes the term  $(S + \frac{1}{T_s})$  is written  $(S - \frac{1}{T_s})$ .

$$\sigma = -\zeta \omega_n$$

$$\omega = \omega_n (1 - \zeta^2)^{1/2}$$

$$\zeta = -\cos \delta$$

Assume that there are two complex pairs of roots that result from the solution of the rigid or equivalent elastic longitudinal characteristic equation. The real-time solution for  $\alpha$  will therefore have the form

$$\alpha = e^{\sigma_p t} \sin (\omega_p t - P_1) + e^{\sigma_{sp} t} \sin (\omega_{sp} t - P_2) \quad (167)$$

where  $P_1$  and  $P_2$  are constants determined by initial conditions.

One pair of roots, the highest frequency pair, determines the short period mode.  
The damped frequency is given by

$$\omega_{sp} \sim \text{radians/second}$$

The period is given by

$$P_{sp} = \frac{2\pi}{\omega_{sp}} \sim \text{seconds/cycle} \quad (168)$$

Time to damp to half amplitude for  $\sigma_{sp} < 0$  is given by

$$T_{1/2} = \frac{1}{\sigma_{sp}} \ln \left( \frac{1}{2} \right) \sim \text{seconds} \quad (169)$$

For  $\sigma_{sp} > 0$ , time to double amplitude is given by

$$T_2 = \frac{1}{\sigma_{sp}} \ln (2) \sim \text{seconds} \quad (170)$$

Cycles to damp to half amplitude is given by

$$C_{1/2} = \frac{T_{1/2}}{P_{sp}} = \frac{\omega_{sp} T_{1/2}}{2\pi} \sim \text{cycles} \quad (171)$$

Cycles to double amplitude is given by

$$C_2 = \frac{T_2}{P_{sp}} = \frac{\omega_{sp} T_2}{2\pi} \sim \text{cycles} \quad (172)$$

These parameters apply also to the longitudinal phugoid mode, the lateral-directional Dutch roll mode and, in some cases, a lateral phugoid mode when the lateral-directional characteristic equation yields two pairs of complex roots.

**8.3.3 Completely elastic airplane.**— For the completely elastic airplane a less restricted approach is required than that used for the rigid and equivalent elastic cases. This involves a mathematical model that will account for the structural dynamic motions of the airplane as well as the flight path and rotational motions. It is accomplished using a model with the option of including or excluding residual flexibility, thus giving a system with an arbitrary number of variables. The airplane has, then, the usual six degrees of freedom plus an arbitrary number of degrees of freedom that involves the structural dynamics.

Subject to the type of problem to be solved and the degree of accuracy required, the engineer has a choice as to the number of variables (degrees of freedom) to include in any one analysis. This is a considerable departure from the philosophy of the well-defined, six rigid-body degrees of freedom associated with the rigid and equivalent elastic mathematical models previously discussed.

The equations of motion that represent both rigid-body and internal motion have been developed as equation (95) in Sec. 5 and are repeated here

$$[a] \{\ddot{q}\} + [B] \{\dot{q}\} + [C] \{q\} = \{0\} \quad (173)$$

Expressions for the coefficients  $[a]$  and  $[B]$  have been presented in Sec. 5 for a residual-flexibility formulation and for the case where all elastic degrees of freedom participate dynamically. For the latter case, i.e., a completely elastic airplane, the coefficient  $[a]$  represents the generalized mass,  $[B]$  includes the aerodynamic damping, and  $[C]$  the generalized stiffness and generalized displacement dependent aerodynamic coefficients.

Taking the Laplace transformation of equation (173) yields

$$[a] s^2 + [B] s + [C] \{q(s)\} = \{0\} \quad (174)$$

NOTE: From the definition of perturbation variables  $\{q\}$ ,

$$\{q(t=0)\} = \{\dot{q}(t=0)\} = \{0\}$$

which will have nontrivial solutions  $\{q\}$  only if

$$|[a] s^2 + [B] s + [C]| = 0 \quad (175)$$



The characteristic equation (175) then yields a determinant with elements  $a_{ij}S^2 + B_{ij}S + C_{ij}$ . When equation (175) is expanded, it yields a polynomial of degree  $\leq 2n$ , where  $n$  is the order of the determinant. The roots of this polynomial are the roots of the characteristic equation (175). These roots are obtained using an eigenvalue approach. The theory is described in refs. 37 and 38.

**8.3.4 Approximate solutions.**— Approximate solutions for rigid airplane frequency and damping characteristics have long been in existence. It is assumed that the appropriate solutions also apply to the equivalent elastic mathematical model because of its similarity to the rigid model.

An extensive discussion of approximate characteristics, transfer functions, etc., can be found in ref. 36. For a two-degree-of-freedom ( $\alpha$  and  $\theta$ ), longitudinal short period mode approximation, the expressions for frequency and damping are:

$$\omega_{n_{sp}} \approx \left[ M_q Z_w - V_{C_1} M_w \right]^{1/2} \quad (176)$$

and

$$\zeta_{sp} \approx \frac{- \left( V_{C_1} M_w + Z_w + M_q \right)}{2\omega_{n_{sp}}} \quad (177)$$

where

$$M_q = \frac{\bar{q} S \bar{c}^2}{2 V_{C_1} I_{yy_1}} C_{m_q}$$

$$Z_w = \frac{\bar{q} S}{M V_{C_1}} \left( -C_{L_\alpha} - C_{D_1} \right) = \frac{\bar{q} S}{M V_{C_1}} C_{Z_\alpha}$$

$$M_w = \frac{\bar{q} S \bar{c}}{V C_1 I_{yy_1}} C_{m_\alpha}$$

$$M_{\dot{w}} = \frac{\bar{q} S \bar{c}^2}{2 V C_1 I_{yy_1}} C_{m_{\dot{\alpha}}}$$

Data in both refs. 4 and 36 show these to be accurate expressions for certain rigid airplanes when compared with the exact quartic solution of equation (166) for longitudinal equations. These expressions have been considered in the light of the study airplanes and the results summarized in table 10. Detailed results may be found in app. C.

The longitudinal phugoid mode can also be approximated by two degrees of freedom,  $u$  and  $\theta$ . Approximate frequency and damping expressions for the phugoid mode are:

$$\omega_{n_P} \approx \left[ -Z_u g / V C_1 \right]^{1/2} \quad (178)$$

and

$$\xi_P \approx -X_u / 2 \omega_{n_P} \quad (179)$$

where

$$Z_u = \frac{\bar{q} S}{M V C_1} \left( -C_{L_u} - 2 C_{L_1} \right) = \frac{\bar{q} S}{M V C_1} C_{Z_u}$$

$$X_u = \frac{\bar{q} S}{M V C_1} \left( -C_{D_u} - 2 C_{D_1} \right) = \frac{\bar{q} S}{M V C_1} C_{X_u}$$

A set of approximate expressions for the lateral-directional modes is given in ref. 36. The Dutch roll, rolling convergence, and spiral modes are given as

$$\omega_{n_D} \approx \left(N_\beta\right)^{1/2} \quad (180)$$

$$\zeta_D \approx \frac{-Y_v - L_P - N_r - \frac{1}{T_S} - \frac{1}{T_r}}{2 \left(N_\beta\right)^{1/2}} \quad (181)$$

(or a less complicated expression

$$\zeta_D \approx \frac{-N_r}{2 \left(N_\beta\right)^{1/2}} \quad (182)$$

which, together with equation (180), gives good results in some cases)

$$\frac{1}{T_r} \approx - \frac{Y_v L_P N_r + L_P N_\beta + \frac{g}{V_{C_1}} L_\beta}{N_\beta} \quad (183)$$

$$\frac{1}{T_s} \approx \frac{\frac{g}{V_{C_1}} \left(N_\beta L_r - L_\beta N_r\right)}{Y_v L_P N_r + L_P N_\beta + \frac{g}{V_{C_1}} L_\beta} \quad (184)$$

where

$$N_{\beta} = \frac{\bar{q} S b}{I_{zz_1}} C_{n_{\beta}}$$

$$Y_v = \frac{\bar{q} S}{M V C_1} C_{y_{\beta}}$$

$$L_P = \frac{\bar{q} S b^2}{2 V C_1 I_{xx_1}} C_{l_P}$$

$$N_r = \frac{\bar{q} S b^2}{2 V C_1 I_{zz_1}} C_{n_r}$$

$$L_r = \frac{\bar{q} S b^2}{2 V C_1 I_{xx_1}} C_{l_r}$$

$$L_{\beta} = \frac{\bar{q} S b}{V C_1 I_{xx_1}} C_{l_{\beta}}$$

The merits of using equations (180) and (182) are discussed in detail in app. C and summarized in table 10. The complexity of equations (183) and (184) makes their use almost ineffectual when compared with the use of the small perturbation program. These expressions for  $1/T_r$  and  $1/T_s$  were not used in this study.

Similarity between the rigid and equivalent elastic mathematical models implies that the use of the approximate expressions could also be used for equivalent elastic cases with some minor redefinitions of terms involved.

Some studies have been made for the completely elastic airplane from the viewpoint of approximate transfer functions. In one study (ref. 39), some approximate frequency and damping expressions were obtained for rigid-body modes with one and two elastic modes for three dissimilar configurations. It is apparent from the study that the form and accuracy of approximate expressions are sensitive to configuration, number of elastic modes being

considered, and dynamic pressure. The treatment of special problems for known significant isolated elastic effects appears to be in the approximation category. The availability, speed, and versatility of digital computer techniques tends to preclude the use of approximate expressions for solving general problems in elastic airplane dynamics.

**8.3.5 Discussion of results of the characteristic equation methods.**— The most important conclusions arrived at from the longitudinal dynamic analyses were that the stability characteristics are more sensitive to aerodynamic derivative accuracy than to elastic effects for the study airplane cases. In addition, the effects of elasticity were relatively small; this is illustrated in fig. 19 for the 707-320B short period frequency and damping characteristics.

Figure 20 shows that the addition of dynamically participating elastic modes to the SST configurations has much less effect on the short period frequency ( $2 \times 2$  versus  $22 \times 22$  modes) than does the static-elastic type of correction ( $3 \times 3$ , rigid versus equivalent elastic). The effect of dynamic pressure is also illustrated in fig. 20 at  $M = 2.7$ . It appears that an increase in  $\bar{q}$  at that condition has an overall stiffening effect, as observed by comparing the elastic increments between the comparable models.

A particularly disturbing quality of the data in fig. 20 is the lack of consistency in the effects of elasticity. For example, the truncated, completely elastic data show increases, decreases, and no changes in the frequency. In addition, the change between rigid and equivalent elastic frequency shows increases in frequency for  $42^\circ$  sweep and decreases at  $72^\circ$  sweep. This precludes guessing or making any general statements as to the overall effects of elasticity. This is even more evident as illustrated in fig. 21 where the undamped natural frequency is presented for the SST. In each case shown, the effect of adding elastic degrees of freedom (generalized coordinates) is illustrated.

In general, the frequency increased when the two lowest frequency elastic modes were added, then decreased when the next sets of two were added out to eight total modes. From there on adding modes had rather unpredictable effects, except that in all cases the frequency tended to approach a constant value as more modes were added beyond 12. The apparent inconsistency is that the constant value is not always less than or more than the rigid  $2 \times 2$ .

The study also showed that adding many elastic degrees of freedom consistently decreased the damping of the short period mode for the study airplanes.

The adequacy of a particular mathematical model (structural or aerodynamic) for the longitudinal dynamics (table 10) would be an important consideration of a handling-qualities study.

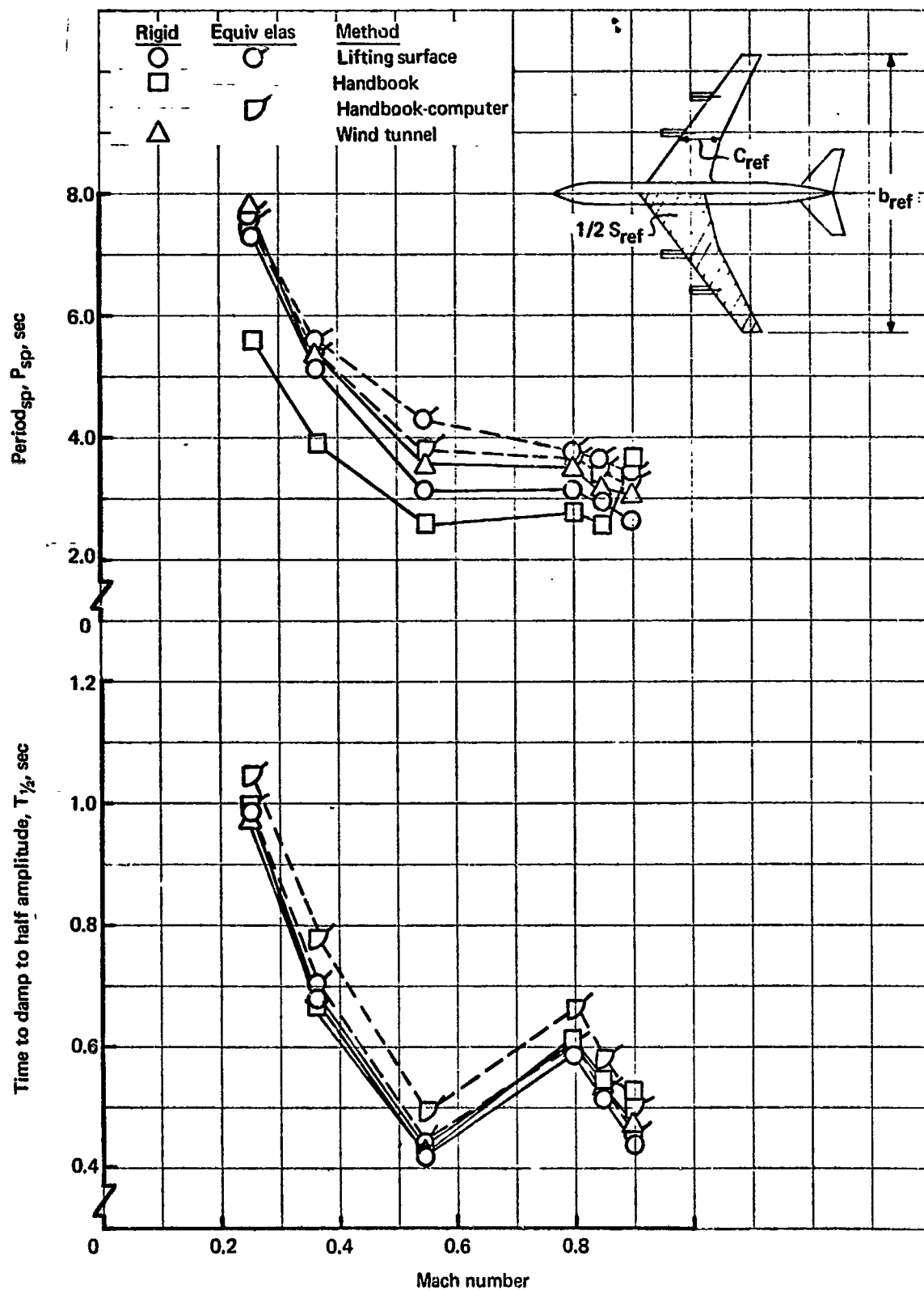


FIGURE 19. - 707-320B LONGITUDINAL SHORT PERIOD RIGID VS. EQUIVALENT ELASTIC

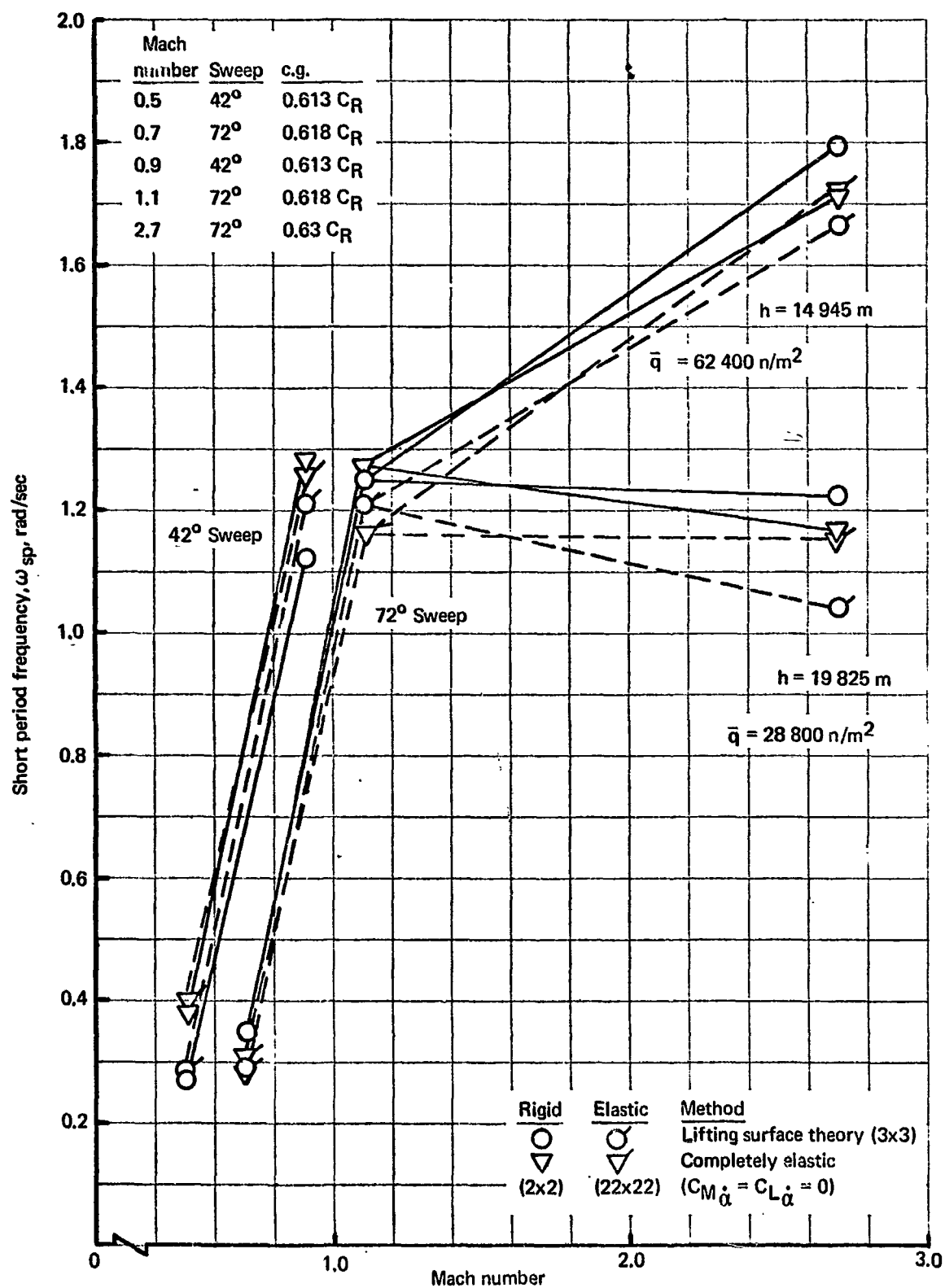


FIGURE 20. – RIGID, EQUIVALENT ELASTIC AND COMPLETELY ELASTIC SHORT FREQUENCY – 42° AND 72° SST

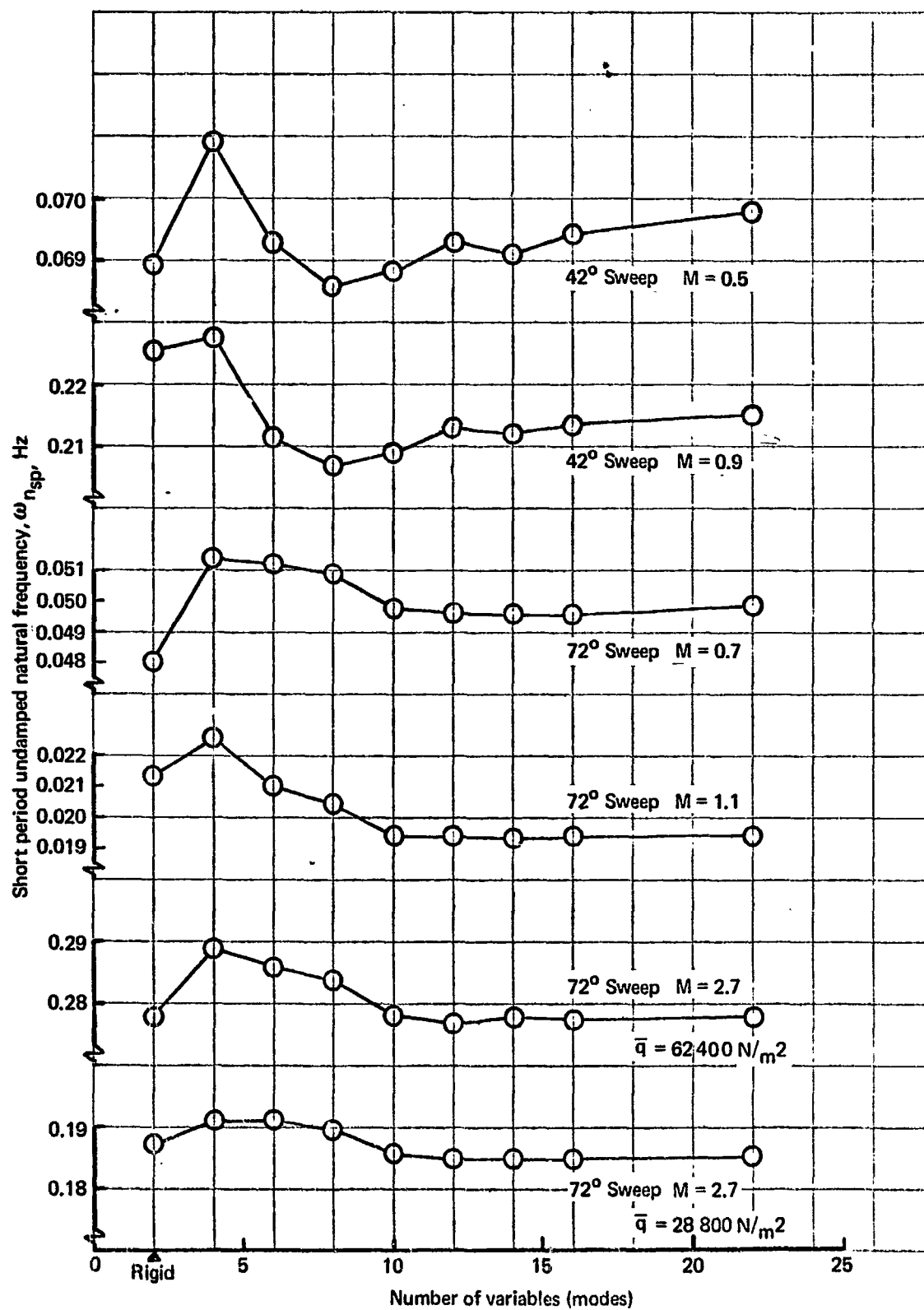


FIGURE 21.— SST,  $\Lambda_{L.E} = 42^\circ$  &  $72^\circ$  VARIATION OF SHORT PERIOD FREQUENCY WITH NUMBER OF MODES



For the study airplanes at the flight conditions analyzed, the elastic effects on the Dutch roll frequency are quite small. This is illustrated in fig. 22 where the undamped natural frequency and damping are shown for two 707-320B flight conditions for various numbers of elastic modes. The effects of residual flexibility are also shown. A static analysis predicts the period of this mode accurately enough for stability and control purposes. The damping of the Dutch roll mode decreases with the addition of the first few elastic modes, but then increases very slightly as more elastic degrees of freedom are added. A static-elastic analysis would appear to predict the damping with sufficient accuracy for this configuration.

The various truncated, completely elastic airplane models gave good correlation with flight test data for the 707-320B for the Dutch roll mode. Figure 23 shows the correlation of the damping. Also shown are the poor results obtained from the equivalent elastic handbook method. The method has certain deficiencies, which are discussed in apps. B and C.

For many cases the variations in dynamic characteristics for the rigid airplane due to the use of different methods for calculating the stability derivatives are as large as any elastic effects (for all modes). This points to the fact that a sophisticated, completely elastic airplane mathematical model is only as good as the basic rigid stability derivatives. (Table 10 summarizes the effects of elasticity on the dynamics for all configurations.) Therefore, a need exists for an accurate analytical approach to generating all stability derivatives in conjunction with lifting surface and lifting line aerodynamic theories. The longitudinal rate derivatives ( $C_{m\dot{\alpha}}$ ,  $C_{m\dot{q}}$ ,  $C_{L\dot{q}}$ , etc.) and all lateral-directional stability derivatives need to be mechanized.

## 8.4 Dynamic Stability Characteristics by Time History Solutions

**8.4.1 Applicability of time history solutions.**— There are today several practical cases where nonlinearities in the equations of motion (dynamic or aerodynamic) are large enough that they cannot be neglected. It has been common practice in such cases to base judgment of stability behavior on time history solutions of the equations of motion. A time history is a set of data that describes airplane motions as a function of real time, i.e.,  $\{X\} = \{X(t)\}$ .

Time histories have the advantage of providing a clear physical picture of the motion of the airplane. In addition, they have the merit of allowing a direct comparison of analytical with experimental data.

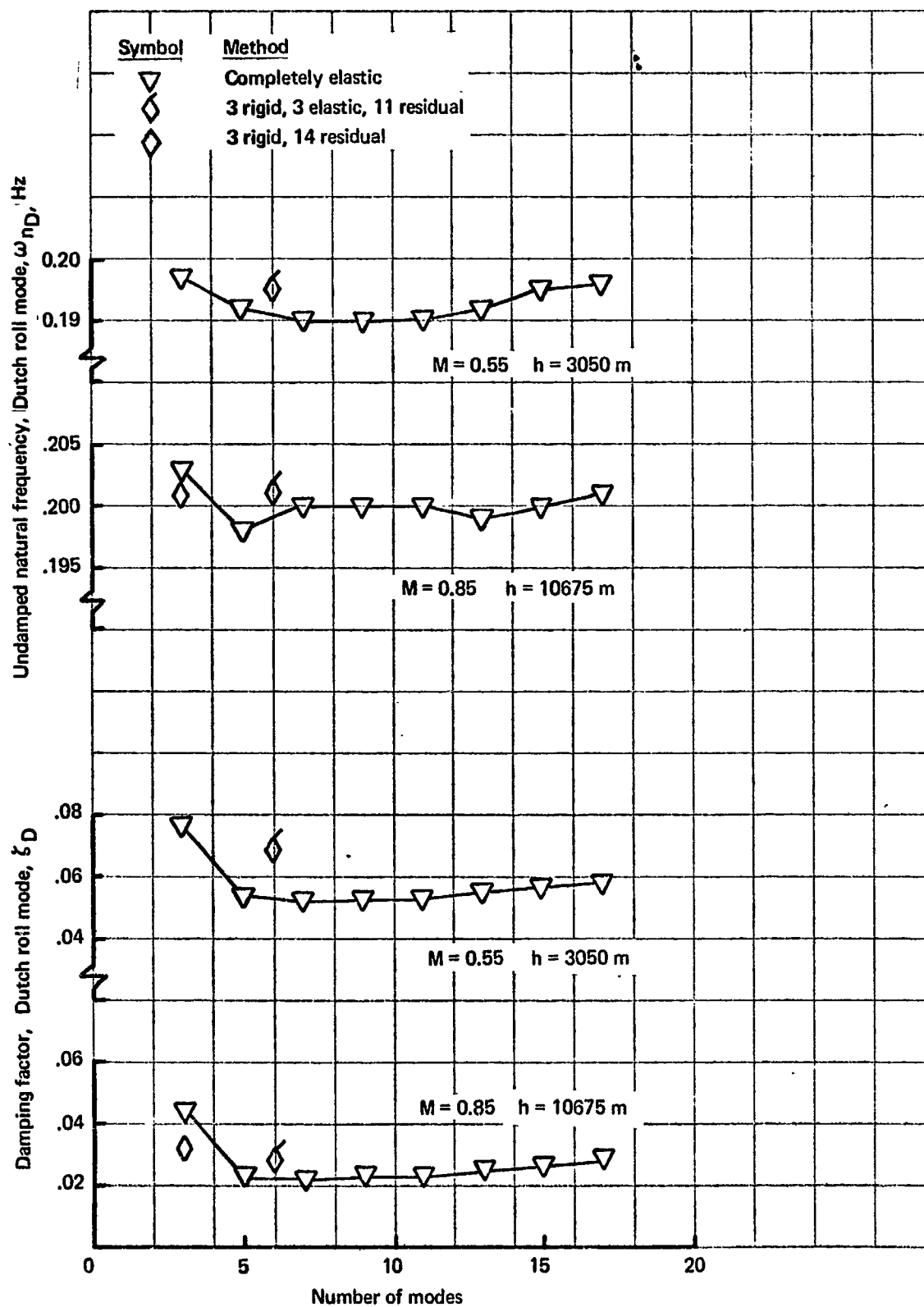


FIGURE 22.— 707-320B DUTCH ROLL FREQUENCY AND DAMPING FOR COMPLETELY ELASTIC AIRPLANE

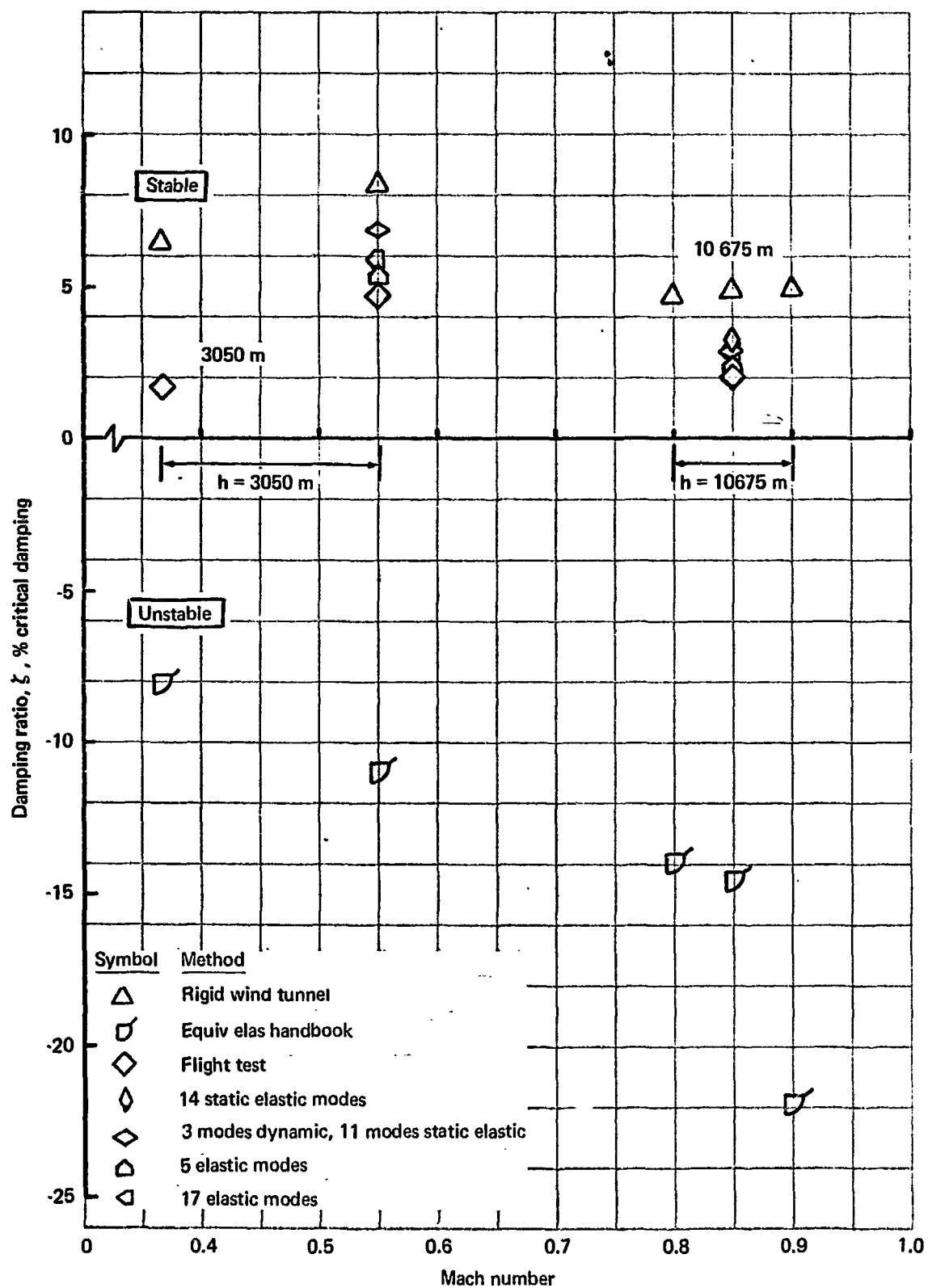


FIGURE 23. - 707-320B DUTCH ROLL DAMPING, ANALYSES RESULTS VS. FLIGHT TEST DATA.

Time histories can be generated by integrating with respect to time the complete airplane equations of motion or, for that matter, any of the equations of motion shown in Sec. 5. The integration technique may vary, but the approach is generally the same for any type of computer. The equations must be trimmed (equilibrated) either separately from or in conjunction with the problem to be solved, i.e., the solutions  $\{X_1\}$  of the algebraic steady-state equations must be obtained and used as initial conditions. The program is started with  $t = 0$ . At some time when  $t_0 \leq 0$ , a disturbance,  $\{\Delta X\}$ , is introduced and the response,  $\{X(t)\}$ , calculated for  $t_0 < t < t_1$ , where  $t_1 - t_0$  is usually a time interval sufficiently long to establish stability behavior but not so long as to involve mass or other changes that would significantly affect assumptions made in deriving the airplane equations of motion.

In this fashion, it is possible to determine stability behavior by observation, i.e., by judging the behavior of the variables of the resulting time history. The stability criteria associated with time histories have been stated in Sec. 6.

An important observation must be made. For nonlinear equations of motion (see app. A), several different cases involving disturbances different in both kind and magnitude must be run to obtain sufficient information to establish the stability behavior. The reason for this must be found in the property of nonlinear differential equations, i.e., that their response behavior can be a function of the initial disturbance.

For the linearized, uncoupled, small perturbation equations of motion (app. A), only one arbitrary disturbance is required for each mode (longitudinal or lateral-directional). Linearity implies that the response behavior is independent of the size or type of disturbance in that mode. However, time history generation for the linear, small perturbation equations is not necessarily the most efficient approach to stability analysis.

The major advantage of the time history (integration) approach is that it is in terms of real time. The analyst has more physical feel for the problem, since he observes motions similar to those which the airplane would experience in flight under the same conditions. Most of the disadvantages of the time history method are not really pertinent to the problem of stability behavior. Instead, they are of an economic nature, involving such things as acquisition, upkeep, and availability of hardware and facilities; and man-hour expenditures in programming, data preparation, and reduction.

**8.4.2 Rigid and equivalent elastic airplanes.**— The time history technique for rigid and equivalent elastic models is essentially that written for the rigid airplane. The mechanized solution will subsequently be referred to as the “rigid-body, six-degree-of-freedom program,” even though it is also used for equivalent elastic solutions. The equations of motion solved by the program, as described in app. A, are the “equations of arbitrary motion.”

For this program these equations can be nonautonomous. Thrust forces and moments may be input as explicit functions of time. In addition, aerodynamic forces and moments due to controls may be explicit functions of time. The aerodynamic data may be nonlinear and aerodynamic cross-coupling may be included, e.g.,  $C_{D\beta}$ ,  $C_{m\beta}$ , and  $C_{L\beta}$ . Certain negligible derivatives such as  $C_{D\dot{\alpha}}$  and  $C_{Y\dot{\delta}_A}$  have been neglected.

A description of the particular integration scheme used to solve the equations of motion for this program may be found in app. C.

The rigid-body, six-degree-of-freedom program is capable of analyzing handling-qualities problems. Part of the basic program output is the velocity ( $U_p$ ,  $V_p$ ,  $W_p$ ) and acceleration ( $\dot{U}_p$ ,  $\dot{V}_p$ ,  $\dot{W}_p$ ) at the pilot's station. Also, because engine thrust may be input separately for each engine as an explicit function of time, the program has the capability to analyze engine-out-type time history solutions.

**8.4.3 Completely elastic airplane.**— The time history solutions of the completely elastic airplane equations of motion for this study were obtained using a special programming language called MIMIC. This special technique is documented in ref. 40. The time histories are merely the time-dependent analogs of the frequency-dependent equation (174) with initial equilibrium conditions to which the perturbations are added along with a disturbance to excite the system. The scheme is simple. From the equilibrium conditions and the disturbance, the accelerations are calculated, e.g.,

$$\ddot{q}_i = \ddot{q}_i \left( \{ \ddot{q} \}, \{ \dot{q} \}, \{ q \} \right) \quad (185)$$

$$i = 1, 2, \dots, n$$

these are integrated by making the statements

$$\begin{aligned} \dot{q}_i &= I N T \left( \ddot{q}_i, \dot{q}_i(0) \right) \\ &= \dot{q}_i(0) + \int \ddot{q}_i dt \quad i = 1, 2, \dots, n \end{aligned} \quad (186)$$

and further by

$$\begin{aligned} q_i &= I N T \left( \dot{q}_i, q_i(0) \right) \\ &= q_i(0) + \int \dot{q}_i dt \quad i = 1, 2, \dots, n \end{aligned} \quad (187)$$

Included as a subroutine in the MIMIC program is a Runge-Kutta numerical integration technique that accomplished the integrations, equations (186) and (187). This approach is easy for the engineer since the programs are then physically oriented. However, as one might expect, the easier the program is to write, for a given problem, the more time is required to execute it.

As was mentioned already, this solution is an analog of the frequency-dependent equation (174). Indeed, aside from errors inherent in the numerical integration technique, there should be no difference between the MIMIC solutions and those obtained from the explicit expression

$$\{q\} = \mathcal{L}^{-1} \left( \left[ [a]s^2 + [B]s + [C] \right]^{-1} \{q_0\} \right) \quad (188)$$

where  $\mathcal{L}^{-1}$  indicates the inverse Laplace transformation.

8.4.4 Discussion of time history solutions.— For the time histories generated for this report, the aerodynamic coefficients were of a linear nature only. Also, the lateral-directional and longitudinal modes were constrained to be uncoupled. The time history methods were used here to obtain a graphic presentation of the motion in response to various disturbances. The frequency and damping characteristics are essentially those given by the characteristic equation method.

An example of the use of the MIMIC program to generate longitudinal time histories is shown in fig. 24. The response of the 707-320B to an elevator pulse is shown for three mathematical models. The rigid model corresponds to the truncated, completely elastic model resulting in the use of a 2 x 2 matrix. The inclusion of 14 dynamic modes in a static-elastic manner results in the greatest pitch amplitudes. If four modes are allowed to participate dynamically, the result is an effective increase in the damping as reflected in the decrease of amplitude.

Appendix C presents more examples of the use of MIMIC and the six-degree-of-freedom program to generate time histories.

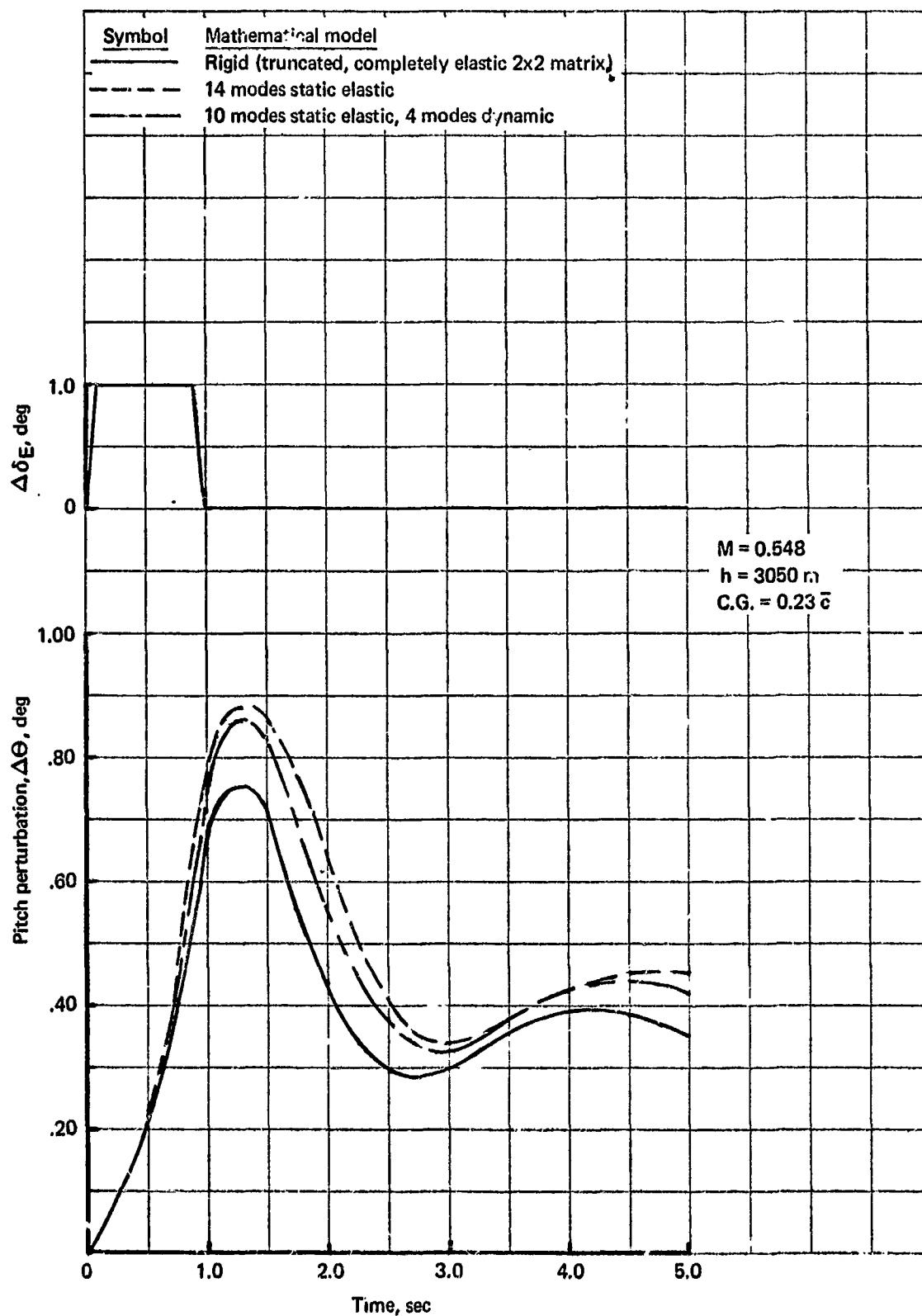


FIGURE 24. - 707-320B PITCH PERTURBATION IN RESPONSE TO ELEVATOR PULSE





## 9.0 SOLUTION OF THE COMPLETE PROBLEM

### 9.1 Arrangement of a Computation System

Previous discussions have dealt with individual parts of the overall problem of determining the stability characteristics of an elastic airplane. The pertinent methods can now be arranged to form a general calculation procedure. Certain preparatory calculations and inputs must also be included as part of the procedure.

To handle elastic airplane problems completely and in sufficient detail for most applications, computer mechanization of the methods is necessary. On this basis, a flow diagram of recommended computer program system is given in fig. 25. This system is now under development.

The airplane definition section computes the characteristics of an airplane in terms of aerodynamic and structural influence coefficients, free vibration normal modes, and inertias. This section will not accept empirical data as input. The airplane stability evaluation section evaluates the stability characteristics of the airplane, using the results from the airplane definition section either alone or in conjunction with empirical data which would apply corrections to or replace the computed results from the airplane definition section.

The airplane definition section consists of four computer programs identified as: geometry definition (GD), aerodynamic influence coefficients (AIC), structural influence coefficients (SIC), and normal modes (NM). The airplane stability evaluation section consists of three computer programs identified as: stability derivatives and static stability (SD&SS), characteristic equation rooting (CER), and time histories (TH). Specifications and comments on each of the program elements are given in following sections. Element specifications were arrived at after careful consideration of the detailed results and developments contained in the appendixes as well as of information in this document.

### 9.2 Use of the System

The system as presented above could be operated in a number of ways to suit the particular needs of the user. It has the capability of accepting very detailed inputs as well as empirical and test data to obtain very accurate answers. It can also be used in a less complex manner to find preliminary-design-type data. The major use classifications can be stated as:

1. Preliminary design;
2. Configuration development;
3. End product development

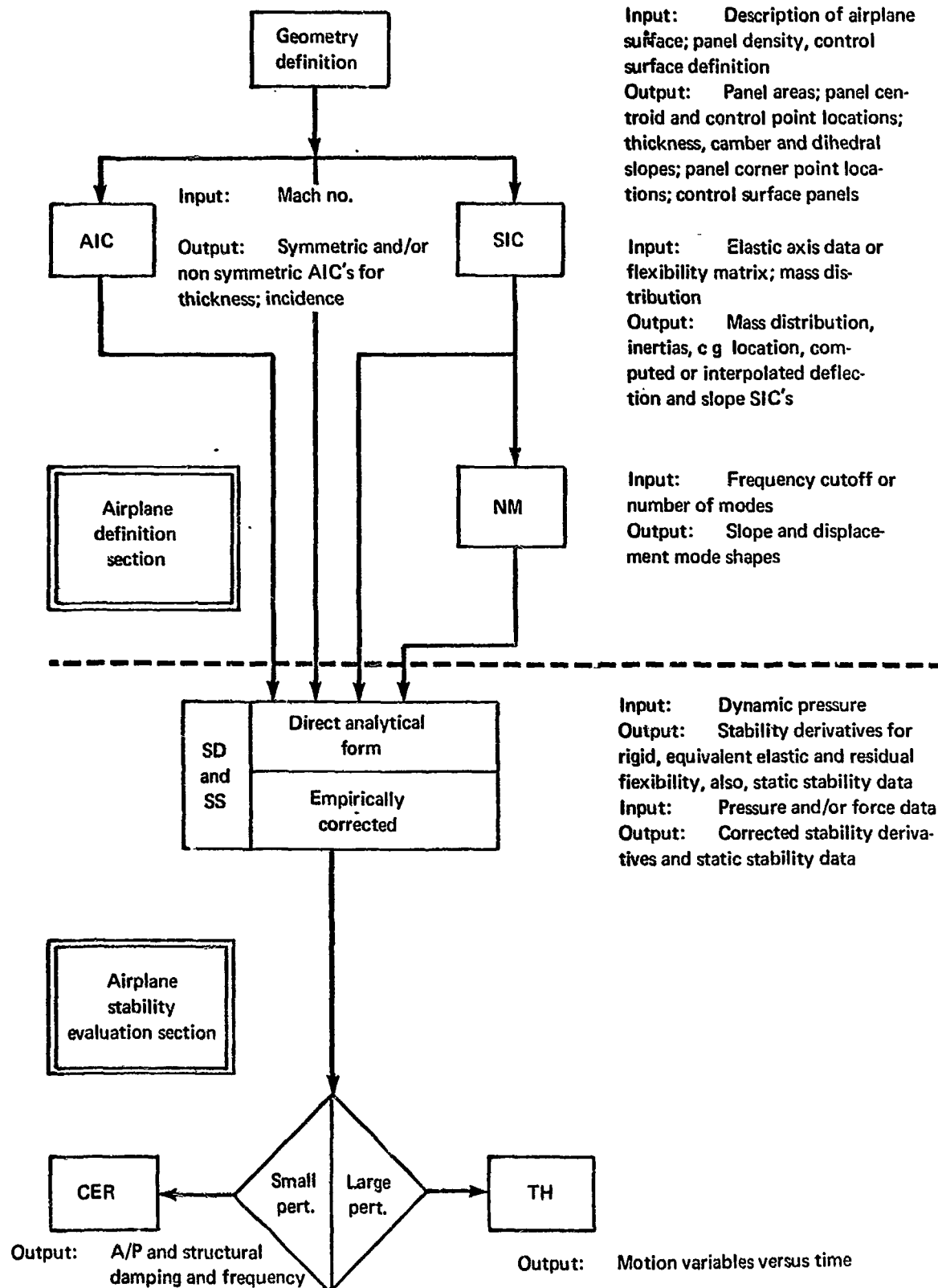


FIGURE 25. — COMPUTING PROGRAM SYSTEM—FLOW DIAGRAM

9.2.1 Preliminary design use.— This category is one for which only relatively rough input data are usually available. Structural characteristics and mass distribution are very approximate, and no wind tunnel or other aerodynamic test data are available. Under these circumstances, coarser airplane paneling would ordinarily be used than for configuration or end product development. It would also be usual to concentrate on finding stability derivatives and static stability characteristics from the SD&SS program. Use would not ordinarily be made of the NM and TH parts of the system nor, possibly, of the CER program.

Therefore, by using the GD, AIC, SIC, SD&SS, and CER elements, preliminary-design-type answers could be obtained on the elastic effects. Comparison of the computed elastic and rigid stability derivatives and static stability would provide a good first look at the role that flexibility plays for a particular configuration. It should be remembered, however, that the specific rigid and elastic values may be in considerable error, although better than obtainable in the past, even when the elastic-to-rigid ratios or increments are adequate.

Use of the system in this manner allows relatively easy calculation of effects due to configuration changes, loading, flight regime changes, etc., for both the rigid and the elastic airplane.

9.2.2 Configuration development use.— For this use, input data for the system are more accurate and complete than for preliminary design. By this time, detailed structural analyses have generally been accomplished, weights determined, and wind tunnel tests run on the basic configuration. Usually aerodynamic force data are available, but not pressure distribution data.

The GD element is now used to give a more accurate description of the airplane. Denser paneling is specified. Details such as dihedral and detailed geometry are included. The AIC program, using the better description, can now calculate more accurate aerodynamic data. Items such as nacelle and wing-body-tail interference, which may have been neglected in the preliminary design studies, are now included.

In the SIC element the early matrix obtained with an approximate beam analysis is replaced with a more exact beam analysis or with an externally developed matrix. If the externally developed matrix is based on different paneling than the AIC's, an interpolation routine in the SIC program changes the matrix to be consistent with the aerodynamic paneling.

It may be elected at this point to include the structural dynamic effect on the stability characteristics. This is initiated by exercising the NM program to provide data for inputting programs in the airplane evaluation section of the system.

In the stability evaluation section, the user has the opportunity to supplement or replace previously calculated data with test, handbook, or empirical data. A modest amount of adjustment is normally desirable for cases where viscous effects and nonlinearities are small. The adjustment would usually take the form of applying calculated elastic-to-rigid ratios to wind-tunnel-obtained stability derivatives. For example,

$$C_{m_{\alpha}}{}_{eq.el.} = \frac{C_{m_{\alpha}}{}_{Elastic}}{C_{m_{\alpha}}{}_{Rigid}} \bigg|_{Computed} \cdot C_{m_{\alpha}}{}_{Rigid} \bigg|_{W.T.}$$

Where viscous effects and/or nonlinearities are important, more extensive adjustment should be made.

The SD&SS program then calculates the derivatives and static stability using the previously calculated data plus desired empirical inputs.

To obtain the dynamic stability characteristics, one chooses either the CER or TH program. The CER program is faster but is more restricted, since it is based on the linear, small perturbation equations. The TH program would be required for cases where large perturbations describe the motion. It can also be used for the small perturbation case, but would not give any more information than could be obtained from the CER program.

The structural dynamic effects can be included in either program by inputting the NM program results. Comparisons of rigid, equivalent elastic, and completely elastic airplane representations are then possible.

**9.2.3 End product development.**— The methods for this area of use are an extension of those of the previous section. More and more test data would be fed into the system so that ultimately the system might be considered as a vehicle for calculating perturbations to the test information. In particular, pressure distribution data, local separations, complete surface stall, viscous wake effects, and other effects would be empirically included. The extent of this type of investigation is usually limited by the time and experience of the user. The main value of this approach is to investigate problem areas that may come to light late in the development of a configuration or in flight test.

### 9.3 System Element Descriptions

**9.3.1 Geometry definition (GD).**— This program accepts as input the basic geometric description of the airplane's surface either in its cruise condition or in its unloaded condition (jig shape). In addition, the program accepts an aerodynamic paneling density selection. Constraints placed on the paneling selection reflect, primarily, the requirements of the aerodynamic representation. The program will have the option of letting the program select the paneling based on a selected density or of operating with a user-selected paneling. The number of aerodynamic panels will be open-ended.

The program will compute thickness, incidence, and dihedral slope at each panel control point and panel centroid, and body surface paneling required for wing-body-tail interference flow on a cylinder of mean body radius. It will also compute the coordinates of the corners, centroids, and control points of the aerodynamic panels as well as their areas.

**9.3.2 Aerodynamic influence coefficients (AIC).**— The surface of the airplane is divided into panels in the geometry definition (GD) program. The function of the AIC program is to compute the change in the pressure force coefficient at each surface panel due to a unit change in inclination to the flow at each panel. The AIC's will include the effects of a slender body, thin wing, and tail as well as wing-body-tail interference. The approach is essentially that of ref. 35. The computation is Mach number dependent and will handle subsonic and supersonic flows up to Mach 5. Transonic flow in the range of about Mach 0.9 to 1.2 is not handled rigorously, so calculations should not be made in the regime. Methods are satisfactory for altitudes up to 30 000 meters (about 100 000 ft).

The AIC's for changes in surface panel inclinations to the flow which are symmetric with respect to the airplane's plane of symmetry are computed separately from those which are nonsymmetric. The method of ref. 35 consists of representing the perturbation of a uniform irrotational flow (due to the presence of the airplane) by line singularities (at inner surfaces of the wing and tail).

Body thickness, camber, and incidence are represented by line sources and line doublets on the body axis. The wing and tail surface thickness slopes are represented by surface distributions of sources with linearly varying strengths. The effects of wing and tail surface incidence and camber, as well as body interference on the wing and tail surfaces, are represented by vorticity distributions.

Unsteady aerodynamic effects can be accounted for in a manner consistent with the requirements of airplane stability evaluation. The method suggested by Miles (ref. 28) for reduced frequencies that are less than unity will be used to reduce the aerodynamics of unsteady flow to one of steady flow.

An aerodynamic influence coefficient theory for nonsymmetrical flow past wing-body-tail combinations has not previously been developed. This development is now being carried out. The techniques involved in the development and theoretical justification are outlined by Van Dyke (ref. 41), Ashley and Landahl (ref. 27), and Chester (ref. 29).

**9.3.3 Structural influence coefficients (SIC).—** This program computes the elements of four flexibility matrices that have the following properties:

- Matrix (1) gives the displacement of each panel control point due to a unit load at each panel centroid.
- Matrix (2) gives the displacement of each panel centroid due to a unit load at each panel centroid.
- Matrix (3) gives the rotation about the y body axis (z body axis for vertical tail and body) at each panel control point due to a unit load at each panel centroid.
- Matrix (4) gives the rotation about the x body axis due to a unit load at each panel centroid.
- All computed flexibility matrices are such that a self-equilibrating system of loads applied at panel centroids does not give rise to a displacement of the airplane's center of gravity or a rigid-body rotation of the airplane about its center of gravity.

The program will also compute the location of the airplane's center of gravity and the components of its inertia in the body axis system.

The program accepts as input:

- Distribution of mass based on paneling computed in GD program.
- Distribution of bending and torsional rigidities along an elastic axis system.
- Geometric description of the elastic axis system.

or

- Elements of a flexibility or a stiffness matrix computed exterior to this system using an arbitrary selection of structural modes.

If elastic axis data are input to the program, the flexibility matrices are computed from beam theory for an arbitrary number of beams. The flexibility matrices are computed by an interpolation method if a flexibility or a stiffness matrix is input to the program. The program is such that an arbitrary number of mass distributions may be combined using multiplying factors.

**9.3.4 Structural normal modes (NM).—** This program is essentially an eigenvalue program. The free vibration of the structure is given by the eigenvalue problem

$$([m] + \omega_n^2 [K]) \{\phi_n\} = 0 \quad (189)$$

The eigenvalues,  $\omega_n$ , are the natural frequencies. The eigenvectors,  $\{\phi_n\}$ , when normalized, are the free vibration normal mode shapes. They give the deflected shapes of the airplane at the resonant natural frequencies,  $\omega_n$ . Slope normal mode shapes can be calculated from the deflection shapes. These results are combined into three mode shape matrices. One relates elastic deflections at the panel control points to generalized coordinates (elastic degrees of freedom), another relates angle-of-incidence changes at the panel control point to elastic degrees of freedom, and the final one relates dihedral changes at the panel control points to elastic degrees of freedom.

**9.3.5 Stability derivative and static stability (SD&SS).—** This program accepts the following as input from the airplane definition section: paneling geometry, normal mode shapes, reference flight condition attitude ( $\theta_1, \phi_1$ ), Mach number, dynamic pressure and load factor, flexibility matrices, aerodynamic influence coefficients, mass distribution, and control surface deflection angles. It will compute all significant stability derivatives based on aerodynamic influence coefficients; as well as control surface trim angles for the reference flight condition, airplane shape in reference flight condition, stick-speed stability, elevator angles per g, neutral point, and maneuver point.

The matrix equations for the stability derivatives formulated for residual flexibility are fundamental to this program. These were presented and discussed in Sec. 7. Recall that the matrices  $[A_1]$ ,  $[A_2]$ ,  $[A_3]$ ,  $[A_4]$ , and  $[A_5]$  appearing in the equations of motion introduced the aerodynamic forces. These matrices, when premultiplied by the matrices  $[\bar{\phi}]^T [B_1]^{-1}$  or  $[\phi_1]^T [B_1]^{-1}$ , result in the airplane stability derivatives for the residual-flexibility formulation. If the matrix  $[K_1]$  in the equation of motion is replaced by a matrix whose elements are all zero, the stability derivatives reduce to the equivalent elastic stability derivatives. These are derived in app. B from the flow boundary condition at the airplane's surface. If, in addition, the flexibility matrix  $[\bar{C}]$  is set equal to a zero matrix, then the stability derivatives are the rigid airplane stability derivatives. Thus, the stability derivatives in the form contained in equation (47) are the most general that may be chosen, since rigid and equivalent elastic derivatives may be readily obtained from them.

Inclusion of a corrector matrix technique in the SD&SS program based on the analysis presented in app. B is recommended. That technique utilizes a diagonal corrector matrix that corrects the aerodynamic influence coefficient matrix using wind tunnel pressure model data. A correction is made at each test condition, e.g., angles of incidence ( $\alpha, \beta$ ) and Mach number. The SD&SS program will accept pressure model data as input. The pressure data cannot be constrained to correspond to the aerodynamic panel centroid selection as they are in the GD program. Therefore, the user is required to select those aerodynamic panel centroids which are to be corrected by each pressure data point. The SD&SS program partitions the aerodynamic influence coefficient matrix in accordance with the user's

selection and performs the corrections on the basis of the partitioned matrix. The program will compute the stability derivatives and static stability results for each test condition for which pressure data is input. The results are then tabulated in a form acceptable as input to the TH program.

The program will also accept as input stability derivative data obtained from wind tunnel and flight testing in tabular form. At the selection of the user, this data would be used in combination with corrected or uncorrected aerodynamic and structural influence coefficients to obtain stability derivatives that are corrected for elastic effects.

The corrections that can be computed are as follows:

- LE/LR = ratio of lift, elastic to rigid
- DE/DR = ratio of drag, elastic to rigid
- YE/YR = ratio of side force, elastic to rigid
- ME/MR = ratio of pitching moment, elastic to rigid
- RE/RR = ratio of rolling moment, elastic to rigid
- NE/NR = ratio of yawing moment, elastic to rigid
- $\Delta ac$  = change in aerodynamic center, elastic from rigid

The program will be such that these elastic-to-rigid corrections can be computed for complete configurations to correct rigid stability derivatives and control derivatives obtained exterior to the program. The program can also compute the elastic-to-rigid corrections for contributions due to the components of a complete configuration, i.e., wing, body, and tail.

**9.3.6 Characteristic Equation Rooting (CER).—** This program operates only on the basis of the small perturbation equations of motion. The stability derivatives will, therefore, be constants and the equations of motion will be linear, ordinary differential equations with constant coefficients.

The program will accept as input the coefficients of the motion variables appearing in the equations of motion. It will combine the coefficients in the appropriate form for the rooting method. The method should be programmed in open-ended form so as to accept as many degrees of freedom as desired.

The program computes:

- times and number of cycles to damp to half and to one-tenth amplitude,
- frequency and period of modes,
- undamped natural frequency of modes,
- damping ratios of modes,
- phase and amplitude of model coupling terms, e.g., argument, and
- $(\phi/\beta)$  and magnitude  $(\phi/\beta)$ .



9.3.7 Time Histories (TH).— This program integrates the large perturbation equations of motion of the airplane by a variable, step-size Runge-Kutta method. The computed motion variables are as follows:

$$\dot{u}, \dot{v}, \dot{w}, u, v, w, V_c = (u^2 + v^2 + w^2)^{1/2}, \dot{p}, \dot{q}, \dot{r}, p, q, r, \Theta, \psi, \phi, n_z \text{ and } h$$

The program will be such that the stability derivatives are input as constants or as variables. When the stability derivatives are variables, their values as functions of the motion variables will be input from tables and interpolated or extrapolated linearly. The interpolation method can handle three independent variables; where more than three are required, superposition will be used to obtain the value of the stability derivative.

The forces and moments from engine thrust will include the effect of engine location and attitude. Inertial effects from rotating engine parts can also be included.

The mass and components of inertia are constant. The airplane will be considered to be flying over a flat, nonrotating earth.



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## 10. CONCLUSIONS AND RECOMMENDATIONS

The conclusions and recommendations listed below have been formulated following an examination of the results of work reported both in this document and in the appendixes.

1. In general, reliable predictions of elastic airplane longitudinal stability and control characteristics can be made using current state-of-the-art theoretical methods. These methods are applicable and practical for preliminary design purposes.
2. The general equations of motion, along with either the large perturbation or small perturbation equations of motion, were found to be suitable for determining reference motion and stability characteristics of an airplane when it is disturbed from the reference motion. The small perturbation equations are applicable to nearly linear systems. The large perturbation equations are necessary in examining nonlinear systems and large disturbances from the reference motion.
3. The mathematical formulation of stability criterion is the same for rigid, equivalent elastic, and completely elastic airplanes.
4. Static stability is usually, but not always, a prerequisite for dynamic stability and good handling qualities.
5. Energy decay methods and Lyapunov stability theory have potential stability criteria application. However, more research is needed to establish their practical application.
6. Handbook methods in some cases give low-quality results for estimation of rigid airplane stability derivatives. In addition, taken alone they cannot adequately predict elastic effects.
7. The aerodynamic influence coefficient method using lifting surface theory gives generally acceptable results for estimation of rigid and equivalent elastic longitudinal derivatives. The method is applicable to lateral-directional problems but has not been mechanized as yet; this should be done.

This method, when arranged for accepting some empirical and test data, was judged to be the best available approach for predicting airplane stability derivatives. Incorporation of leading edge suction, more accurate shed vortex field representation, and other improvements are possible and should be developed where greater accuracy is desired than would be obtained with the basic method.

8. If wind tunnel data are available, the method of (7) above can be used to provide more accurate predictions of equivalent elastic effects. This is done by computing an elastic-to-rigid ratio or increment, which is then applied to the wind tunnel value. For example,

$$C_{m\alpha_{Eq.El.}} = \frac{C_{m\alpha_{Elastic}}}{C_{m\alpha_{Rigid}}} \bigg|_{\text{Computed}} \cdot C_{m\alpha_{Rigid}} \bigg|_{\text{W. T.}}$$

This approach is recommended as a practical and reasonably accurate way to obtain equivalent elastic effects.

9. The remarks in (6), (7), and (8) above also apply to prediction of airplane static stability.
10. Both static and dynamic stability characteristics were found to be sensitive to inaccuracies in estimating the rigid stability derivatives. This uncertainty is usually as large as the elastic effects.
11. Dynamic flexibility effects were usually found to be modest and smaller than the equivalent elastic effects. About 20 elastic modes are required to obtain good results; using less than this number can lead to significant error if residual flexibility is not used.
12. For many cases the equivalent elastic formulation represents the airplane accurately enough so that there is no need to go to the extra complication of the completely elastic formulation. Engineering judgment is required to decide on this for dynamic stability evaluation of any particular configuration, however.
13. Approximate formulas for determining damping of longitudinal dynamics were satisfactory provided  $C_{m\alpha}$  is dominant. However, these methods are unreliable for general use and should be avoided.
14. Characteristic equation rooting and time history methods are adequate and are recommended for use in determining both longitudinal and lateral-directional dynamic stability characteristics.

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